Sobolev Spaces and Partial Differential Equations

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0 Notations

Throughout this book, we assume that U is an open subset of $\mathbb{R}^n$. Given a function $u : U \to \mathbb{R}$, we write $u(x) = u(x_1, \cdots, x_n)$ for $x \in U$. For $i \in [n]$, we write

$$\partial_{x_i} u(x) = \frac{\partial u}{\partial x_i}(x) = u_{x_i}(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h}, \quad x \in U$$

for the partial derivative with respect to variable x_i, provided the limit exists. Partial derivatives of higher orders are similarly defined. If $u : U \to \mathbb{R}$ is twice differentiable, we write $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$ and $\nabla^2 u : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ for its *gradient* and *Hessian matrix*, respectively:

$$\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \cdots, \frac{\partial u}{\partial x_n}(x) \right), \qquad \nabla^2 u(x) = \begin{pmatrix} u_{x_1 x_1}(x) & u_{x_1 x_2}(x) & \cdots & u_{x_1 x_n}(x) \\ u_{x_2 x_1}(x) & u_{x_2 x_2}(x) & \cdots & u_{x_2 x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_n x_1}(x) & u_{x_n x_2}(x) & \cdots & u_{x_n x_n}(x) \end{pmatrix}.$$

The Laplacian Δu of u is defined as the trace of Hessian matrix:

$$\Delta u(x) = \mathrm{tr}(\nabla^2 u(x)) = \frac{\partial^2 u}{\partial x_1^2}(x) + \cdots + \frac{\partial^2 u}{\partial x_n^2}(x).$$

Now we introduce the multi-index notation. A vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ consists of nonnegative integers is called a *multi-index of order* $|\alpha| = \alpha_1 + \cdots + \alpha_n$, Given this multi-index α, we define

$$\partial^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha^n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha^n} u(x).$$

If K is a nonnegative integer, we write

$$\partial^k u(x) := \{\partial^\alpha u(x) : |\alpha| = k\}$$

for the set of all partial derivatives of order k, and define

$$\|\partial^k u\|_{L^p(U)} = \left(\sum_{\alpha : |\alpha| = k} \|\partial^\alpha u\|_{L^p(U)}^p \right)^{1/p}, \quad |\partial^k u| = \|\partial^k u\|_{L^2(U)} = \left(\sum_{\alpha : |\alpha| = k} |\partial^\alpha u|^2 \right)^{1/2}.$$

Furthermore, we replace the symbol ∂ by D when we refer to weak derivatives:

$$\int_U u \partial^\alpha \phi \, dm = (-1)^{|\alpha|} \int_U (D^\alpha u) \phi \, dm, \quad \forall \phi \in C_c^\infty(U),$$

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}, \ \|D^k u\|_{L^p(U)} = \left(\sum_{\alpha : |\alpha| = k} \|D^\alpha u\|_{L^p(U)}^p \right)^{1/p}, \ |D^k u| = \left(\sum_{\alpha : |\alpha| = k} |D^\alpha u|^2 \right)^{1/2}.$$

We use D and D^2 to denote the gradient and Hessian matrix in weak sense:

$$Du(x) = (D_{x_1}(x), \cdots, D_{x_n}(x)), \qquad D^2 u(x) = \begin{pmatrix} u_{x_1 x_1}(x) & u_{x_1 x_2}(x) & \cdots & u_{x_1 x_n}(x) \\ u_{x_2 x_1}(x) & u_{x_2 x_2}(x) & \cdots & u_{x_2 x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_n x_1}(x) & u_{x_n x_2}(x) & \cdots & u_{x_n x_n}(x) \end{pmatrix}.$$

1 Convolution and Smoothing

1.1 Convolution

In this section we first deal with functions on $\mathbb{R}^n$. If a function f is defined on $U \subset \mathbb{R}^n$, we can replace it by its natural zero extension $f : \mathbb{R}^n \to \mathbb{R}$ which assigns $f(x) = 0$ for $x \notin U$.

Definition 1.1 (Convolution)**.** Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions. Define the bad set as

$$E(f, g) := \left\{ x \in \mathbb{R}^n : \int_{\mathbb{R}^n} |f(x-y)g(y)| \, dy = \infty \right\}.$$

The *convolution* of f and g is the function $f * g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$(f * g)(x) = \begin{cases} \int_{\mathbb{R}^n} f(x-y)g(y) \, dy, & x \notin E(f, g), \\ 0, & x \in E(f, g). \end{cases}$$

Remark. Define $F : \mathbb{R}^{2n} \to \mathbb{R}, (x, y) \mapsto f(x)$ and $G : \mathbb{R}^{2n} \to \mathbb{R}, (x, y) \mapsto g(y)$. Then both F and G are measurable functions on $\mathbb{R}^{2n}$, as well as their product $F \cdot G : (x, y) \mapsto f(x)g(y)$. Given linear transformation $T(x, y) = (x - y, y)$, the composition $H = (F \cdot G) \circ T : (x, y) \mapsto f(x-y)g(y)$ is measurable. By Tonelli's theorem, the function $x \mapsto \int_{\mathbb{R}^n} |H(x, y)| \, dy$ is measurable, and $E(f, g)$ is a Lebesgue measurable set.

Clearly, the convolution operation is both commutative and associative, i.e. $f * g = g * f$, and $(f * g) * h = f * (g * h)$. Furthermore, the distributivity of convolution with respect to functional addition immediately follows, i.e. $f * (g + h) = f * g + f * h$.

Proposition 1.2 (Properties of convolution)**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions.*

(i) If $f, g \in L^1(\mathbb{R}^n)$, then the bad set $E(f, g)$ is of measure zero. Moreover, $f * g \in L^1(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^m} (f * g) \, dm = \int_{\mathbb{R}^n} f \, dm \int_{\mathbb{R}^n} g \, dm. \tag{1.1}$$

(ii) If $f \in C_0(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in C_0(\mathbb{R}^n)$.

(iii) If $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f * g \in L^p(\mathbb{R}^n)$, and

$$\|f * g\|_p \leq \|f\|_p \|g\|_1.$$

Proof. (i) Define the measurable function $H(x, y) \mapsto f(x-y)g(y)$ on $\mathbb{R}^{2n}$. By Tonelli's theorem,

$$\int_{\mathbb{R}^{2n}} |H| \, dm = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)| \, |g(y)| \, dx \right) dy = \|f\|_1 \|g\|_1.$$

Hence $H : \mathbb{R}^{2n} \to \mathbb{R}$ is integrable. By Fubini's theorem, for a.e. $x \in \mathbb{R}^n$, $y \mapsto H(x, y)$ is integrable, hence $m(E(f, g)) = 0$. Furthermore, the function $f * g : x \mapsto \int_{\mathbb{R}^n} H(x, y) \, dy$ is also integrable, that is, $f * g \in L^1(\mathbb{R}^n)$. The equation (1.1) follows from Fubini's theorem.

(ii) Given $\epsilon > 0$. By uniform continuity of f, there exists $\eta > 0$ such that $|f(x) - f(x')| < \epsilon / \|g\|_1$ for all $|x - x'| < \eta$, . As a result, for all $x, x' \in \mathbb{R}^n$ such that $|x - x'| < \eta$, we have

$$|(f * g)(x) - (f * g)(x')| \leq \int_{\mathbb{R}^n} |f(x-y) - f(x'-y)| \, |g(y)| \, dy < \epsilon.$$

(iii) is a special case of the following proposition. □

Proposition 1.3 (Young's convolution inequality)**.** *Given $r \in [1,\infty]$ and Hölder r-conjugates $p,q \in [1,\infty]$, i.e. $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then the bad set $E(f,g)$ is of measure zero, and we have*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Remark. Note that

$$r = \frac{pq}{p+q-pq} \geq 1 \quad \Leftrightarrow \quad \frac{pq}{p+q} \geq \frac{1}{2} \quad \Leftrightarrow \quad p \geq \frac{q}{2q-1} \quad \Leftrightarrow \quad q \geq \frac{p}{2p-1},$$

and

$$r < \infty \quad \Leftrightarrow \quad p+q > pq \quad \Leftrightarrow \quad p < \frac{q}{q-1} \quad \Leftrightarrow \quad q < \frac{p}{p-1}.$$

Proof. We first bound $f * g$. By applying generalized Hölder's inequality on $\frac{1}{r}+\frac{r-p}{pr}+\frac{r-q}{qr}=1$, we have

$$\begin{aligned}
|(f*g)(x)| &\leq \int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\;dy = \int_{\mathbb{R}^n} \left(|f(x-y)|^p|g(y)|^q\right)^{1/r} |f(x-y)|^{\frac{r-p}{r}} |g(y)|^{\frac{r-q}{r}}\;dy \\
&\leq \left(\int_{\mathbb{R}^n} |f(x-y)|^p|g(y)|^q\;dy\right)^{1/r} \left(\int_{\mathbb{R}^n} |f(x-y)|^p\;dy\right)^{\frac{r-p}{pr}} \left(\int_{\mathbb{R}^n} |g(y)|^q\;dy\right)^{\frac{r-q}{qr}} \\
&= \left(\int_{\mathbb{R}^n} |f(x-y)|^p|g(y)|^q\;dy\right)^{1/r} \|f\|_p^{\frac{r-p}{r}} \, \|g\|_q^{\frac{r-q}{r}}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\;dy\right)^r dx &\leq \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|^p|g(y)|^q\,dy\,dx\right) \|f\|_p^{r-p}\,\|g\|_q^{r-q} \\
&\leq \|f\|_p^{r-p}\,\|g\|_q^{r-q} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^p\,dx\right) |g(y)|^q\,dy = \|f\|_p^r\,\|g\|_q^r,
\end{aligned}$$

where we use Fubini's theorem in the last inequality. Hence $m(E(f,g))=0$, and $\|f*g\|_r \leq \|f\|_p\|g\|_q$. ☐

Remark. If $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, and $g \in L^q(\mathbb{R}^n)$ is compactly supported, then $f*g \in L^r_{\mathrm{loc}}(\mathbb{R}^n)$.

Review: Compact supported functions. Let X be a topological space. The support of function $f : X \to \mathbb{R}$ is defined as the closure of the set of all points in X not mapped to zero by f:

$$\operatorname{supp} f = \overline{\{x \in X : f(x) \neq 0\}} = \overline{\{f \neq 0\}}.$$

If the support of f is compact in X, f is said to be *compactly supported.* Following this definition, any function defined on a closed interval $[a,b]$ can be extended to a compactly supported function on $\mathbb{R}$.

The set of all continuous compactly supported functions on X is denoted by $C_c(X)$. If $f \in C_c(X)$, then f is uniformly continuous on $\operatorname{supp} f$. Note that $f=0$ outside $\operatorname{supp} f$, we have that f is uniformly continuous on X, which implies $C_c(X) \subset C_0(X)$. Furthermore, by extreme value theorem, f has maximum and minimum on $\operatorname{supp} f$, which implies that f is uniformly bounded on X, i.e. $\max_{x\in X}|f(x)| < \infty$.

Let $(X,\mathscr{A},\mu)$ be a measure space where X is a topological space. Following the discussion above, we have $C_c^\infty(X) \subset L^\infty(X,\mathscr{A},\mu)$ since every $f \in C_c^\infty(X)$ satisfies $\|f\|_\infty = \max_{x\in X}|f(x)| \leq \infty$. Furthermore, if every compact set in X has finite measure, i.e. $\mu(K) < \infty$ for all compact $K \subset X$, then the compactly supported function are always p-integrable:

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{1/p} = \left(\int_{\operatorname{supp} f} |f|^p dm\right)^{1/p} \leq \mu(\operatorname{supp} f)^{1/p}\|f\|_\infty < \infty.$$

Proposition 1.4 (Convolution of compactly supported functions)**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$.*

(i) If $f, g \in L^1(\mathbb{R}^n)$, then $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g} := \overline{\{x + y : x \in \operatorname{supp} f, y \in \operatorname{supp} g\}}$. Furthermore, if both f and g are compactly supported on $\mathbb{R}$, then $f * g$ is also compactly supported. In this case, $\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g$.

(ii) Let $1 \leq p \leq \infty$, and let $k \in \mathbb{N}_0$. If $f \in C_c^k(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $f * g \in C_0^k(\mathbb{R}^n)$. Furthermore, differentiation commutes with convolution, i.e.,

$$\partial^\alpha (f * g) = \partial^\alpha f * g, \qquad \forall |\alpha| \leq k,$$

(iii) Let $1 \leq p \leq \infty$. If $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $f * g \in C_0^\infty(\mathbb{R}^n)$. Similarly, differentiation commutes with convolution, i.e., $\partial^\alpha(f * g) = \partial^\alpha f * g$ for multi-indices α.

Remark. Combining (i) and (ii)/(iii), we obtain a useful conclusion. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. If $f \in C_c^k(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ is compactly supported, then $f * g \in C_c^k(\mathbb{R}^n)$.

Proof. (i) Let $f, g \in L^1(\mathbb{R}^n)$, and take any $x \in \mathbb{R}^n$. Then

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y)\, dy = \int_{(x - \operatorname{supp} f) \cap \operatorname{supp} g} f(x - y) g(y)\, dy.$$

For $x \notin \operatorname{supp} f + \operatorname{supp} g$, we have $(x - \operatorname{supp} f) \cap \operatorname{supp} g = \emptyset$, which implies $(f * g)(x) = 0$. Hence

$$(f * g)(x) \neq 0 \ \Rightarrow x \in \operatorname{supp} f + \operatorname{supp} g \ \Rightarrow \ \operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}.$$

If $f, g \in C_c(\mathbb{R}^n)$, then $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact in $\mathbb{R}^n$. Define $\phi(x, y) = x + y$, which is a continuous map on $\mathbb{R}^{2n}$. Then $\operatorname{supp} f + \operatorname{supp} g = \phi(\operatorname{supp} f \times \operatorname{supp} g)$ is also compact. Consequently, $\operatorname{supp} f + \operatorname{supp} g$ is closed, and its closed subset $\operatorname{supp}(f * g)$ is also compact. which implies $f * g \in C_c(\mathbb{R}^n)$.

(ii) *Step I:* We first show the case $k = 0$. Let $q = p/(p-1)$. Note that f is continuous and compact supported, then $m(\operatorname{supp} f) < \infty$, f is uniformly continuous, and $\|f\|_\infty = \max_{x \in \operatorname{supp} f} |f(x)| < \infty$. By Hölder's inequality, for all $x \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} |f(x - y)|\, |g(y)|\, dy \leq \|f\|_q \|g\|_p \leq m\big(\operatorname{supp} f\big)^{1/q} \|f\|_\infty \|g\|_p < \infty.$$

Then $f * g$ is well-defined on $\mathbb{R}^n$. To show uniform continuity of $f * g$, we fix $\epsilon > 0$ and let η be such that $|x - x'| < \eta$ implies $|f(x) - f(x')| < \epsilon$. Then

$$\begin{aligned} |(f * g)(x) - (f * g)(x')| &= \left| \int_{\mathbb{R}^n} [f(x - y) - f(x' - y)]\, g(y)\, dy \right| \\ &\leq 2m\big(\operatorname{supp} f\big)^{1/q} \|g\|_p\, \epsilon. \end{aligned}$$

Step II: We prove the case $k = 1$. It suffices to show the interchangeability of derivative and integral. Given any quantity $h > 0$, we have

$$\frac{(f * g)(x + he_i) - (f * g)(x)}{h} = \int_{\mathbb{R}^n} \frac{f(x + he_i - y) - f(x - y)}{h} g(y)\, dy. \tag{1.2}$$

Since $f \in C_c^1(\mathbb{R}^n)$, by Lagrange's mean value theorem, there exists $\xi \in [0, 1]$ such that

$$\left| \frac{f(x + he_i - y) - f(x - y)}{h} \right| = \left| \partial_{x_i} f(x + \xi h e_i - y) \right|, \tag{1.3}$$

Note that $\partial_{x_i} f$ is also continuous and compactly supported on $\mathbb{R}^n$, the RHS of (1.3) is bounded by $\|\partial_{x_i} f\|_\infty$, and the integrand in (1.2) is dominated by an integrable function $\|\partial_{x_i} f\|_\infty g$. Using Lebesgue's dominate convergence theorem, we have

$$\lim_{h\to 0}\int_{\mathbb{R}^n}\frac{f(x+he_i-y)-f(x-y)}{h}g(y)\,dy=\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_i}(x-y)g(y)\,dy.$$

Therefore $\partial_{x_i}(f*g)=\partial_{x_i}f*g$. Since $\partial_{x_i}f\in C_c(\mathbb{R}^n)$, we have $\partial_{x_i}(f*g)\in C_0(\mathbb{R}^n)$, and $f*g\in C_0^1(\mathbb{R}^n)$.

Step III: Use induction. Suppose our conclusion holds for $C_c^{k-1}(\mathbb{R}^n)$. For each $f\in C_c^k(\mathbb{R}^n)\subset C_c^{k-1}(\mathbb{R}^n)$, $\partial^{k-1}f\subset C_c^1(\mathbb{R}^n)$. By Step II, for any $|\alpha|=k-1$,

$$\partial^{\alpha+e_i}(f*g)=\partial_{x_i}(\partial^\alpha(f*g))=\partial_{x_i}(\partial^\alpha f*g)=(\partial^{\alpha+e_i}f)*g,$$

which is uniformly continuous on $\mathbb{R}^n$. Hence $f*g\in C_0^k(\mathbb{R}^n)$.

(iii) Note that $C_c^\infty(\mathbb{R}^n)=\bigcap_{k=0}^\infty C_c^k(\mathbb{R}^n)$, we have $\partial^\alpha(f*g)=\partial^\alpha f*g$ for all $\alpha\in\mathbb{N}_0^n$. Following Step II, $\partial^\alpha f\in C_c(\mathbb{R}^n)$ implies $\partial^\alpha(f*g)\in C_0(\mathbb{R}^n)$ for all $\alpha\in\mathbb{N}_0^n$. Hence $f*g\in\bigcap_{k=0}^\infty C_0^k(\mathbb{R}^n)=C_0^\infty(\mathbb{R}^n)$. $\square$

Review: Translation operators. Let X be a vector space, let Y^X be the set of functions $f:X\to Y$, and let s be a vector in X. The *translation operator* $\tau_s:Y^X\to Y^X$ is defined as

$$(\tau_s f)(x)=f(x-s),\ \forall f\in Y^X.$$

Proposition 1.5. *Let $1\le p<\infty$. For any $f\in C_c(\mathbb{R}^n)$,*

$$\lim_{s\to 0}\|\tau_s f-f\|_p=0. \tag{1.4}$$

Proof. Let $f\in C_c(\mathbb{R}^n)$, and let B_1 be the closed unit ball in $\mathbb{R}^n$. The collection of functions $\{\tau_s f:|s|\le 1\}$ has a common support

$$K=\bigcup_{|s|\le 1}\operatorname{supp}(\tau_s f)=\operatorname{supp}f+B_1=\{x+y:x\in\operatorname{supp}f,y\in B_1\}=\phi(\operatorname{supp}f\times B_1),$$

which is compact as the image of a compact set under a continuous map $\phi:\mathbb{R}^{2n}\to\mathbb{R}^n,(x,y)\mapsto x+y$.

By uniform continuity of f, given $\epsilon>0$, there exists $\delta>0$ such that $|f(x)-f(y)|<\epsilon$ for all $|x-y|<\delta$. Then for any s with $|s|<|\min(\delta,1)|$, we have

$$\|\tau_s f-f\|_p^p=\int_K|f(x-s)-f(x)|^p dx\le\mu(K)\,\epsilon^p.$$

Since $\mu(K)<\infty$, and ϵ is arbitrary, we conclude that $\|\tau_s f-f\|_p\to 0$ as $s\to 0$. $\square$

Review: Mollifier. A *mollifier* on $\mathbb{R}^n$ is a symmetric function $\eta\in C_c^\infty(\mathbb{R}^n)$ supported on the closed unit ball $B_1=\{x\in\mathbb{R}^n:|x|\le 1\}$ with $\int_{\mathbb{R}^n}\eta\,dm=1$. For example, the *standard mollifier* is defined as

$$\eta(x)=\frac{1}{Z}\exp\left(\frac{1}{|x|^2-1}\right)\chi_{B_1}(x),\quad \textit{where}\ \ Z=\int_{|t|\le 1}\exp\left(\frac{1}{|t|^2-1}\right)\,dt.$$

For each $\epsilon>0$, we set

$$\eta_\epsilon(x)=\frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)\quad\Rightarrow\quad\int_{\mathbb{R}^n}\eta_\epsilon(x)\,dx=1,\ \operatorname{supp}(\eta_\epsilon)\subset B(0,\epsilon).$$

Now we provide an important approximation result using compactly supported smooth functions.

Proposition 1.6. *For $1 \le p < \infty$, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*

Proof. Let $f \in C_c(\mathbb{R}^n)$. We choose a mollifier $\eta \in C_c^\infty(\mathbb{R}^n)$, and define $\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right)$ for $\epsilon > 0$. By Proposition 1.4, $f * \eta_\epsilon \in C_c^\infty(\mathbb{R}^n)$, and

$$\begin{aligned}
\int_{\mathbb{R}^n} |(f*\eta_\epsilon)(x) - f(x)|^p \, dx &= \int_{\mathbb{R}^n} \left| \int_{|y|\le\epsilon} (f(x-y) - f(x))\eta_\epsilon(y)\, dy \right|^p dx \\
&\le \int_{\mathbb{R}^n} \int_{|y|\le\epsilon} |f(x-y) - f(x)|^p \, \eta_\epsilon(y)\, dydx \qquad \text{(By Jensen's inequality)} \\
&= \int_{|y|\le\epsilon} \eta_\epsilon(y) \|\tau_y f - f\|_p^p \, dy \\
&\le \sup_{y:|y|\le\epsilon} \|\tau_y f - f\|_p^p.
\end{aligned}$$

which converges to 0 as $\epsilon \to 0$ by Proposition 1.5. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, the result follows. □

Application I: continuity of translation operators in L^p-spaces. The limit (1.4) in Proposition 1.5 remains zero for all $f \in L^p(\mathbb{R})$. We fix $\epsilon > 0$, so there exists $g \in C_c^\infty(\mathbb{R})$ such that $\|f - g\|_\infty < \epsilon/3$ by Proposition 1.6. Choose δ such that $\|\tau_s g - g\|_p < \epsilon/3$ for all $|s| < \delta$. Then for all $|s| < \delta$,

$$\|\tau_s f - f\|_p \le \|\tau_s f - \tau_s g\|_p + \|\tau_s g - g\|_p + \|g - f\|_p = 2\|f - g\| + \|\tau_s g - g\|_p < \epsilon.$$

Application II: uniform continuity of convolution. Let $\frac{1}{p} + \frac{1}{q} = 1$ be Hölder conjugates. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in C_0(\mathbb{R}^n)$. Given $\epsilon > 0$, we choose $\delta > 0$ such that $\|\tau_s f - f\|_p < \epsilon/\|g\|_q$ for all $|s| \le \delta$. Then one have

$$|(f*g)(x-s) - (f*g)(x)| \le \int_{\mathbb{R}^n} |f(x-s-y) - f(x-y)|\, |g(y)|\, dy \le \|\tau_s f - f\|_p \|g\|_q < \epsilon$$

for all $x \in \mathbb{R}^n$ and all $|s| < \delta$. Clearly, $f * g$ is uniformly continuous on $\mathbb{R}^n$.

Application III: uniform continuity of convolution on bounded sets. If $f \in L^p(\mathbb{R}^n)$ is compactly supported, and $g \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, we have $f * g \in C(\mathbb{R}^n)$. We fix $\epsilon > 0$ and $R > 0$, choose $r > 0$ such that $\operatorname{supp} f \subset B(0,r)$, and choose $\delta > 0$ such that $\|\tau_s f - f\|_p < \epsilon/\|g\chi_{B(0,R+r)}\|_q$ for all $|s| < \delta$. Then

$$|(f*g)(x) - (f*g)(x')| \le \int_{B(0,R+r)} |f(x-y) - f(x'-y)|\, |g(y)|\, dy \le \|\tau_{x-x'} f - f\|_p \|g\chi_{B(0,R+r)}\|_q < \epsilon$$

for all $|x|, |x'| < R$ with $|x - x'| < \delta$. Hence $f * g$ is uniformly continuous on the open ball $O(0,R)$.

In addition, if $f \in C_c^\infty(\mathbb{R}^n)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we have $f * g \in C^\infty(\mathbb{R}^n)$. This result can be shown by adapting the proof of Proposition 1.4.

1.2 Local Mollification

In this section we study the approximation of locally integrable functions. Our discussion is based on a bounded open region $U \subset \mathbb{R}^n$. Given any $\epsilon > 0$, we define

$$U^\epsilon = \{x \in U : d(x, \partial U) > \epsilon\}.$$

Since U is open, U^ϵ is nonempty for sufficiently small $\epsilon > 0$. In addition, the continuity of $d(\cdot, \partial U)$ implies that U^ϵ is also an open region. Furthermore, given any precompact open set $V \Subset U$, since $d(\overline{V}, \partial U) > 0$, we can find $\epsilon > 0$ such that $V \Subset U^\epsilon \Subset U$.

Definition 1.7 (Mollification)**.** Given $u \in L^1_{\text{loc}}(U)$, define its *mollification* by

$$u^\epsilon := \eta_\epsilon * u,$$

where we abuse the notation u in this expression to denote the zero extension of $u : U \to \mathbb{R}$ on $\mathbb{R}^n$. The value of this mollification in U^ϵ is given by

$$u^\epsilon(x) = \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)\,dy = \int_{B(0,1)} \eta(z)u(x+\epsilon z)\,dz. \tag{1.5}$$

Remark. The mollification u^ϵ is smooth in U^ϵ. For any $x \in U^\epsilon$, we take $\delta > 0$ such that $B(x,\delta) \Subset U^\epsilon$. Then $u^\epsilon = \eta_\epsilon * \chi_{B(x,\epsilon+\delta)} u$ in $B(x,\delta)$. Since $u \in L^1_{\text{loc}}(U)$, by Proposition 1.4 (iii), u^ϵ is infinitely continuously differentiable at x. Note that differentiability is a local property, we conclude that $u^\epsilon \in C^\infty(U^\epsilon)$.

Proposition 1.8 (Properties of mollification)**.** *Let $u \in L^1_{\text{loc}}(U)$.*

(i) $u^\epsilon \to u$ a.e. on U as $\epsilon \downarrow 0$.

(ii) If $u \in C(U)$, then $u^\epsilon \to u$ uniformly on compact subsets of U.

(iii) If $1 \le p < \infty$ and $u \in L^p_{\text{loc}}(U)$, then $u^\epsilon \to u$ in $L^p_{\text{loc}}(U)$.

Proof. (i) According to Lebesgue's differentiation theorem, we have

$$\lim_{r\downarrow 0} \frac{1}{r^n} \int_{B(x,r)} |u(y) - u(x)|\,dy = 0$$

for a.e. $x \in U$. Since $x \in U^\epsilon$ for sufficiently small $\epsilon > 0$, we have

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} |u^\epsilon(x) - u(x)| &\le \lim_{\epsilon\downarrow 0} \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)-u(x)|\,dy \\
&\le \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \|\eta\|_\infty |u(y)-u(x)|\,dy = 0, \quad \text{for a.e. } x \in U.
\end{aligned}$$

Consequently, we have $u^\epsilon \to u$ *a.e.* on U as $\epsilon \downarrow 0$.

(ii) Given a compact $K \subset U$, we choose $\delta > 0$ sufficiently small such that $K \subset U^\delta$. Since u is a continuous function, the bad set $E(\eta_\epsilon, u)$ is empty. Then for all $\epsilon \in (0,\delta]$, one have

$$\begin{aligned}
\sup_{x\in K} |u^\epsilon(x) - u(x)| &= \sup_{x\in K} \left| \int_{B(0,1)} \eta(z)\left(u(x+\epsilon z) - u(x)\right) dz \right| \\
&\le \sup_{x\in K} \sup_{z\in B(0,1)} |u(x+\epsilon z) - u(x)|
\end{aligned}$$

Since $x, x+\epsilon z \in U^\delta$, we have $|u(x+\epsilon z) - u(x)| \rightrightarrows 0$ by uniform continuity of u on $\overline{U}^\delta$.

(iii) Given any pre-compact set $V \Subset U$, we first choose a pre-compact subset W of U such that $V \Subset W \Subset U$. We claim that, for sufficiently small $\epsilon > 0$, we have $\|u^\epsilon\|_{L^p(V)} \le \|u\|_{L^p(W)}$. To this end, we note that

$$\begin{aligned}|u^\epsilon(x)| = \left|\int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)\,dy\right| &\le \int_{B(x,\epsilon)} \eta_\epsilon(x-y)^{1-1/p}\eta_\epsilon(x-y)^{1/p}|u(y)|\,dy \\ &\le \underbrace{\left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)\,dy\right)}_{=1}^{1-1/p}\left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)|^p\,dy\right)^{1/p}.\end{aligned}$$

We choose $\epsilon > 0$ such that $V \Subset W^\epsilon$. Then

$$\|u^\epsilon\|_{L^p(V)}^p \le \int_V \left(\int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u(y)|^p\,dy\right)dx \le \int_W \left(\int_{B(y,\epsilon)} \eta_\epsilon(x-y)\,dx\right)|u(y)|^p\,dy = \|u\|_{L^p(W)}^p.$$

Now we fix $\delta > 0$, and choose $g \in C(W)$ such that $\|f-g\|_{L^p(W)} < \delta/2$. Then

$$\begin{aligned}\|f^\epsilon - f\|_{L^p(V)} &\le \|f^\epsilon - g^\epsilon\|_{L^p(V)} + \|g^\epsilon - g\|_{L^p(V)} + \|g-f\|_{L^p(V)} \\ &\le \|g^\epsilon - g\|_{L^p(V)} + 2\|g-f\|_{L^p(W)} \le \|g^\epsilon - g\|_{L^p(V)} + \delta.\end{aligned}$$

By (ii), $g^\epsilon \rightrightarrows g$ on V as $\epsilon \downarrow 0$, hence $\limsup_{\epsilon\downarrow 0}\|f^\epsilon - f\|_{L^p(V)} \le \delta$. $\square$

Remark. If U is bounded and $u \in L^p(U)$, we can extend u to $\mathbb{R}^n$ to conclude that $u^\epsilon \to u$ in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$. Since $U \Subset \mathbb{R}^n$, we have $u' \to u$ in $L^p(U)$.

Now we provide an application of mollification.

Lemma 1.9. *If $v \in L^1_{\mathrm{loc}}(U)$, and*

$$\int_U v\phi\,dm = 0 \quad \forall \phi \in C_c^\infty(U), \tag{1.6}$$

then $v = 0$ a.e..

Proof. Let K be a compact subset of U, and choose $\varphi \in C_c^\infty(U)$ such that $0 \le \varphi \le 1$, and $\varphi \equiv 1$ on K. [We will show the existence of such function in Lemma 1.10.] By assumption (1.5), we have

$$(\eta_\epsilon * v_\varphi)(x) = \int_{\mathbb{R}^n} \eta_\epsilon(x-y)\varphi(y)v(y)\,dy = \int_U \underbrace{\eta_\epsilon(x-y)\varphi(y)}_{\phi_{\epsilon,x}(y)} v(y)\,dy = 0,$$

since $\phi_{\epsilon,x}(\cdot) = \eta_\epsilon(x-\cdot)\varphi(\cdot) \in C_c^\infty(U)$. By letting $\epsilon \to 0$, we obtain the limit $\eta_\epsilon * v_\varphi \xrightarrow{L_1} \varphi v = 0$ a.e.. Consequently, we have $v = 0$ a.e. on all compact subsets K of U.

Define $K_r = \{x \in \mathbb{R}^n : d(x, U^c) \ge 2/r \text{ and } |x| \le r\}$. Then $K_r \subset U$ is compact, and $U = \bigcup_{r=1}^\infty K_r$. Since $v = 0$ a.e. on all K_r, we have

$$m(\{v = 0\}) = m\left(\bigcup_{r=1}^\infty K_r \cap \{v = 0\}\right) = \lim_{r\to\infty} m(K_r \cap \{v=0\}) = 0.$$

Hence $v = 0$ *a.e.* on U. $\square$

Remark. Due to the property (1.5), the functions in the class $C_c^\infty(U)$ of compactly supported smooth functions are also called *test functions.*

1.3 Application: Smooth Partition of Unity

In this, section we employ the mollification approach to construct partitions of unity. These technical results are later used to obtain global properties from local ones.

Lemma 1.10 (C^∞-Urysohn lemma). *Let U be an open subset of $\mathbb{R}^n$, and let K be a compact subset of U. Then there exists a function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on K, and $\operatorname{supp}\varphi \subset U$.*

Proof. Given $\epsilon > 0$, we define

$$K_\epsilon := \{x \in \mathbb{R}^n : d(x, K) \le \epsilon\}.$$

Choose $\epsilon > 0$ so small that $K_{2\epsilon} \subset U$, and let $\varphi = \eta_\epsilon * \chi_{K_\epsilon}$. By properties of convolution, $\varphi \in C_c^\infty(\mathbb{R}^n)$, $0 \le \varphi \le 1$, and $\varphi \equiv 1$ on K. Moreover, $\operatorname{supp}\varphi \subset \overline{\operatorname{supp}\eta_\epsilon + K_\epsilon} \subset K_{2\epsilon} \subset U$, $\square$

Next we introduce a technical lemma in topology, which asserts that we are able to "shrink" a finite open cover of a closed subset of $\mathbb{R}^n$.

Lemma 1.11. *Let $U \subset \mathbb{R}^n$, and let $\{U_i\}_{i=1}^N$ be a collection of open subsets of $\mathbb{R}^n$ such that $\overline{U} \subset \bigcup_{i=1}^N U_i$. Then there exists a collection $\{V_i\}_{i=1}^N$ of open subsets of $\mathbb{R}^n$ such that $\overline{V_i} \subset U_i,\ i = 1, \cdots, N$ and $\overline{U} \subset \bigcup_{i=1}^N V_i$.*

Proof. We proceed by substituting the elements of the cover of $\overline{U}$ one by one. Let $A_1 = \overline{U} \backslash (U_2 \cup \cdots \cup U_N)$. Then A_1 is a closed set contained in U_1. By normality of $\mathbb{R}^n$, we can choose an open set V_1 containing A_1 such that $\overline{V}_1 \subset U_1$. Then we obtain a cover $\{V_1, U_2, \cdots, U_N\}$ of $\overline{U}$.

At the k^{th} step, we are given open sets $V_1, \cdots, V_{k-1}$ such that $\{V_1, \cdots, V_{k-1}, U_k, \cdots, U_N\}$ covers U. We let $A_k = \overline{U} \backslash (V_1 \cup \cdots \cup V_{k-1} \cup U_{k+1} \cup \cdots \cup U_N)$, and choose an open set V_k such that $A_k \subset V_k \subset \overline{V}_k \subset U_k$. Then $\{V_1, \cdots, V_k, U_{k+1}, \cdots, U_N\}$ is also an open cover of $\overline{U}$. At the n^{th} step, our result is proved. $\square$

Remark. In addition, if U is bounded, we may assume that each U_i is bounded. As a result, we can obtain a shrunk open cover $\{V_i\}_{i=1}^N$ of $\overline{U}$ such that $V_i \Subset U_i$. In other words, each $\overline{V}_i$ is a compact set.

Theorem 1.12 (Partition of unity). *Let U be a bounded, open subset of $\mathbb{R}^n$, and let $(V_i)_{i=1}^N$ be a collection of open sets in $\mathbb{R}^n$ such that $U \Subset \bigcup_{i=1}^N V_i$. Then there exists a family of smooth functions $(\psi_i)_{i=1}^N : \mathbb{R}^n \to [0,1]$ such that $\psi_i \in C_c^\infty(V_i)$ for all $i = 1, \cdots, N$, and $\sum_{i=1}^N \psi_i \equiv 1$ on U.*

Remark. The family $(\psi_i)_{i=1}^N$ is called a *smooth partition of unity subordinate to the open sets $(V_i)_{i=1}^N$*.

Proof. By Lemma 1.11, we take a collection $(K_i)_{i=1}^N$ of compact subsets of $\mathbb{R}^n$ such that $K_i \subset V_i,\ i = 1, \cdots, N$ and $\overline{U} \subset \bigcup_{i=1}^N K_i$. By Lemma 1.10, for each $i = 1, \cdots, N$, there exists a smooth function $\varphi_i : \mathbb{R}^n \to [0,1]$ such that $\varphi \equiv 1$ on K_i, and $\operatorname{supp}\varphi_i \subset V_i$. We then define

$$\psi_1 = \varphi_1,\ \psi_2 = (1 - \varphi_1)\varphi_2,\ \cdots,\ \psi_N = (1 - \varphi_1)\cdots(1 - \varphi_{N-1})\varphi_N.$$

Then $0 \le \psi_i \le 1$, and $\psi_i \in C_c^\infty(V_i)$ for all $i = 1, \cdots, N$. Furthermore,

$$1 - \sum_{i=1}^N \psi_i = (1 - \varphi_1)(1 - \varphi_2)\cdots(1 - \varphi_N).$$

For each point $x \in U \subset \bigcup_{i=1}^N K_i$, at least one factor $(1 - \varphi_i)$ vanishes, and we have $\sum_{i=1}^N \psi_i \equiv 1$ on U. $\square$

2 Sobolev Spaces

2.1 Hölder Spaces

Assume that $U \subset \mathbb{R}^n$ is open and $\gamma \in (0,1]$. A function $u : U \to \mathbb{R}$ is said to be *Hölder continuous with exponent* γ, if there exists some constant $C > 0$ such that

$$|u(x) - u(y)| \leq C|x-y|^\gamma, \quad \forall x, y \in U.$$

In this section, we first discuss the Hölder spaces, which contain functions with some nice properties.

Definition 2.1 (Hölder spaces)**.** Let $U \subset \mathbb{R}^n$ be open, and $0 < \gamma \leq 1$. If $u : U \to \mathbb{R}$ is a bounded and Hölder continuous function, we define

$$\|u\|_{C(\overline{U})} := \sup_{x\in U} |u(x)|, \quad [u]_{C^{0,\gamma}(\overline{U})} = \sup_{x,y\in U: x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma},$$

where $[\cdot]_{C^{0,\gamma}(\overline{U})}$ is the γ^{th}*-Hölder seminorm.* The γ^{th}*-Hölder norm* is defined as

$$\|u\|_{C^{0,\gamma}(\overline{U})} = \|u\|_{C(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}.$$

Let $k \in \mathbb{N}_0$. The *Hölder space* $C^{k,\gamma}(\overline{U})$ consists of all functions $u \in C^k(\overline{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\overline{U})} := \sum_{\alpha:|\alpha|\leq k} \|\partial^\alpha u\|_{C(\overline{U})} + \sum_{\alpha:|\alpha|=k} [\partial^\alpha u]_{C^{0,\gamma}(\overline{U})}$$

is finite. In other words, $C^{k,\gamma}(\overline{U})$ contains all k-times continuously differentiable functions whose k^{th}-partial derivatives are bounded and Hölder continuous with exponent γ.

Remark. One can easily check that $C^{k,\gamma}(\overline{U})$ is a vector space, and $\|\cdot\|_{C^{k,\gamma}(\overline{U})}$ is a norm on $C^{k,\gamma}(\overline{U})$.

Theorem 2.2. *The Hölder space $C^{k,\gamma}(\overline{U})$ is a Banach space.*

Proof. It suffices to show completeness of $C^{k,\gamma}(\overline{U})$ under the norm $\|\cdot\| = \|\cdot\|_{C^{k,\gamma}(\overline{U})}$. Let (u_l) be a Cauchy sequence in $C^{k,\gamma}(\overline{U})$, i.e. $\|u_l - u_m\| \to 0$ as $i, j \to \infty$. By completeness of $C(\overline{U})$, (u_l) converges uniformly to some $u \in C(\overline{U})$, and for each $|\alpha| \leq k$, the sequence $(\partial^\alpha u_l)$ converges uniformly to some function $u^{(\alpha)} \in C(\overline{U})$. Consequently, we have $\partial^\alpha u_l \to \partial^\alpha u = u^{(\alpha)}$ for all $|\alpha| \leq k$, and $u \in C^k(\overline{U})$.

Now it remains to discuss Hölder continuity. Since (u_l) is a Cauchy sequence, there exists $M > 0$ such that $\sup_{l\in\mathbb{N}} \|u_l\| \leq M$. For all $|\alpha| = k$,

$$\frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^\gamma} \leq \frac{|\partial^\alpha u(x) - \partial^\alpha u_l(x)|}{|x-y|^\gamma} + \underbrace{\frac{|\partial^\alpha u_l(x) - \partial^\alpha u_l(y)|}{|x-y|^\gamma}}_{\leq M} + \frac{|\partial^\alpha u_l(y) - \partial^\alpha u(y)|}{|x-y|^\gamma}.$$

Since $\partial^\alpha u_l \rightrightarrows \partial^\alpha u$, the first and third terms in the last display converges to zero for all $x, y \in U$. Hence $\partial^\alpha u$ is Hölder continuous with exponent γ. Furthermore,

$$\frac{|\partial^\alpha(u_l - u)(x) - \partial^\alpha(u_l-u)(y)|}{|x-y|^\gamma} = \lim_{m\to\infty} \frac{|\partial^\alpha(u_l-u_m)(x) - \partial^\alpha(u_l-u_m)(y)|}{|x-y|^\gamma} \leq \lim_{m\to\infty} [\partial^\alpha(u_l-u_m)]_{C^{0,\gamma}(\overline{U})}$$

Since the last bound does not depend on $x, y \in U$, we can obtain $[\partial^\alpha(u_l - u)]_{C^{0,\gamma}(\overline{U})} \to 0$ by letting $l \to \infty$. Hence $\|u_l - u\| \to 0$ as $l \to \infty$. ☐

2.2 Weak Derivatives

Review: Integration by Parts. Let $U \subset \mathbb{R}^n$ be a open and bounded region with C^1 boundary. According to the divergence theorem, for each vector field $\mathbf{u} \in C^1(\overline{U}, \mathbb{R}^n)$, we have

$$\int_U (\nabla \cdot \mathbf{u})\, dx = \int_{\partial U} \mathbf{u} \cdot \nu \, dS,$$

where $\nu : \partial\Omega \to \mathbb{R}^n$ is the outward pointing normal vector field. For $u \in C^1(\overline{U})$, we set $\mathbf{u} = ue_i$. Then

$$\int_U \frac{\partial u}{\partial x_i}\, dx = \int_{\partial U} u\nu^i \, dS, \quad i = 1, \cdots, n.$$

Now assume we are given a function $u \in C^1(U)$. If $\phi \in C^\infty(U)$, we apply the above formula to $u\phi$:

$$\int_U u \frac{\partial \phi}{\partial x_i}\, dx = -\int_U \frac{\partial u}{\partial x_i} \phi \, dx, \quad i = 1, \cdots, n.$$

More generally, if $k \in \mathbb{N}$, $u \in C^k(U)$, and α is a multi-index with $|\alpha| = k$, then

$$\int_U u(\partial^\alpha \phi)\, dx = (-1)^{|\alpha|} \int_U (\partial^\alpha u)\phi \, dx.$$

This formula gives rise to the definition of weak derivatives.

Definition 2.3 (Weak derivatives)**.** Assume that $u, v \in L^1_{\mathrm{loc}}(U)$ and α is a multi-index. Then v is said to be the α^{th}*-weak partial derivative of* u, written $\partial^\alpha u = v$, if

$$\int_U u\partial^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v\phi \, dx.$$

for all test functions $\phi \in C_c^\infty(U)$.

Remark. Suppose both v and $\widetilde{v}$ are α^{th}-weak partial derivatives of u. By applying Lemma 1.9 on $v - \widetilde{v}$, one can show that the α^{th}-weak partial derivative of u is uniquely defined up to a set of measure zero. Note that the weak derivatives are only a.e. determined.

Example 2.4. Consider the function $u(x) = |x|$, which is in $L^1_{\mathrm{loc}}(\mathbb{R})$. Then the weak derivative of u on $\mathbb{R}$ is

$$v(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

Now we verify this claim. Given any test functions $\phi \in C_c^\infty(\mathbb{R})$, let $\operatorname{supp}\phi \subset [-M, M]$. Then we have

$$\begin{aligned}\int_{\mathbb{R}} u(x)\phi'(x)\, dx &= \int_0^M x \, d\phi(x) - \int_{-M}^0 x \, d\phi(x) \\ &= -\int_0^M \phi(x)\, dx + \int_{-M}^0 \phi(x)\, dx = -\int_{\mathbb{R}} v(x)\phi(x)\, dx.\end{aligned}$$

However, the function $v \in L^1_{\mathrm{loc}}(\mathbb{R})$ has no weak derivative. We argue by contradiction, and assume that there exists $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} v(x)\phi'(x)\, dx = -\int_{\mathbb{R}} w(x)\phi(x)\, dx, \quad \forall \phi \in C_c^\infty(\mathbb{R}).$$

Then we have

$$\int_{\mathbb{R}} w(x)\phi(x)\,dx = -\int_{\mathbb{R}} v(x)\phi'(x)\,dx = -\int_0^\infty \phi'(x)\,dx + \int_{-\infty}^0 \phi'(x)\,dx = 2\phi(0).$$

Now we choose a sequence $\phi_m(x) = \exp\left(\frac{1}{|mx|^2-1}\right)\chi_{(-\frac{1}{m},\frac{1}{m})}$ in $C_c'(\mathbb{R})$, which satisfies $\phi_m \to \mathrm{e}^{-1}\chi_{\{0\}}$. If we replace ϕ by ϕ_m in the last display and let $m \to \infty$, the LHS and RHS converges to different values, a contradiction! Hence v is not weakly differentiable.

Now we discuss the equivalence of weak and partial derivatives of differentiable functions.

Lemma 2.5. *Suppose a continuous function $u : U \to \mathbb{R}$ is weakly differentiable, and the weak derivatives $D^{e_1}u, \cdots, D^{e_n}u$ are also continuous (thus unique). Then $u \in C^1(U)$, and the weak derivatives coincide with the partial ones, in symbols $(\partial^{e_1}u, \cdots, \partial^{e_n}u) = (D^{e_1}u, \cdots, D^{e_n}u)$.*

Proof. Since differentiation is a local problem, we fix any pre-compact set $V \Subset U$ and choose $\epsilon > 0$ such that $V \subset U^\epsilon$. Then the value of the mollification u^ϵ inside U^ϵ is given by (1.6). For each $x \in U^\epsilon$, we have

$$\begin{aligned}
(\partial^{e_i}u^\epsilon)(x) = (\partial^{e_i}\eta_\epsilon * u)(x) &= \int_{B(x,\epsilon)} (\partial_x^{e_i}\eta_\epsilon)(x-y)u(y)\,dy \\
&= -\int_{B(x,\epsilon)} (\partial_y^{e_i}\eta_\epsilon)(x-y)u(y)\,dy \\
&= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)(D^{e_i}u)(y)\,dy = \left(\eta_\epsilon * D^{e_i}u\right)(x).
\end{aligned}$$

By Proposition 1.8, $\epsilon \downarrow 0$ gives uniform convergences $u^\epsilon \rightrightarrows u$ and $\partial^{e_i}u^\delta = \eta_\epsilon * D^{e_i}u \rightrightarrows D^{e_i}u$ on the compact set $\overline{V}$. Moreover, for any $x \in V$ and any $|h| > 0$ such that $x + he_i \in V$,

$$\begin{aligned}
u(x+he_i) - u(x) &= \lim_{\epsilon\downarrow 0} \left(u^\epsilon(x+he_i) - u^\epsilon(x)\right) \\
&= \lim_{\epsilon\downarrow 0} \int_0^h (\partial^{e_i}u^\epsilon)(x+te_i)\,dt = \int_0^h (D^{e_i}u)(x+te_i)\,dt.
\end{aligned}$$

By continuity of $D^{e_i}u$, we have $\partial_{e_i}u(x) = D^{e_i}u(x)$ for all $x \in V$. Hence $u \in C^1(V)$. Since the pre-compact set V is arbitrary, we have $u \in C^1(U)$. $\square$

Remark. In fact, this proof also provide an approximation approach of weak derivatives. If a function $u : U \to \mathbb{R}$ has weak derivative $D^\alpha u$, we choose any $V \Subset W \Subset U^\epsilon$. Then

$$\begin{aligned}
(\partial^\alpha u^\epsilon)(x) = (\partial^\alpha\eta_\epsilon * u)(x) &= \int_{B(x,\epsilon)} (\partial_x^\alpha\eta_\epsilon)(x-y)u(y)\,dy \\
&= (-1)^{|\alpha|}\int_{B(x,\epsilon)} (\partial_y^\alpha\eta_\epsilon)(x-y)u(y)\,dy \\
&= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)(D^\alpha u)(y)\,dy = \left(\eta_\epsilon * D^\alpha u\right)(x).
\end{aligned}$$

Hence $\partial^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u = (D^\alpha u)^\epsilon$ on $W \subset U^\epsilon$. Since $D^\alpha u \in L^1_{\mathrm{loc}}(U) \subset L^1_{\mathrm{loc}}(W)$, by Proposition 1.8, $\partial^\alpha u^\epsilon \to D^\alpha u$ in $L^1(V)$ as $\epsilon \to 0$. Furthermore, since $V \Subset U$ is arbitrary, we have

$$\partial^\alpha u^\epsilon \to D^\alpha u \quad in \quad L^1_{\mathrm{loc}}(U) \quad as \quad \epsilon \to 0.$$

This result also gives rise to the following approximation theorem.

Theorem 2.6 (Characterization of weak derivatives)**.** *A function $u \in L^1_{\text{loc}}(U)$ is weakly differentiable in U if and only if there is a sequence of functions $u_m \in C^\infty(U)$ such that $u_m \to u$ and $\partial^\alpha u_m \to v$ in $L^1_{\text{loc}}(U)$. In that case the weak derivative of u is given by $v = D^\alpha u \in L^1_{\text{loc}}(U)$.*

Proof. If u is weakly differentiable in U, we can construct a desired sequence by mollification, as is discussed in the preceding Remark. Conversely, given such a sequence (u_m), we have

$$\left|\int_U u_m\phi\,dm - \int_U u\phi\,dm\right| = \left|\int_{\operatorname{supp}\phi}(u_m - u)\phi\,dm\right| \le \|\phi\|_\infty \int_{\operatorname{supp}\phi}|u_m - u|\,dm \to 0, \quad \forall \phi \in C_c^\infty(U).$$

Consequently, the L^1_{loc}-convergence of u_m and $\partial^\alpha u_m$ implies

$$\int_U u\partial^\alpha\phi\,dm = \lim_{n\to\infty}\int_U u_m\partial^\alpha\phi\,dm = \lim_{n\to\infty}(-1)^{|\alpha|}\int_U(\partial^\alpha u_m)\phi\,dm = (-1)^{|\alpha|}\int_U v\phi\,dm.$$

Therefore, u is weakly differentiable, and $v = D^\alpha u$. ☐

Next we introduce some properties of weak derivatives. Many results from the classical differential calculus may be extended to weak derivatives.

Proposition 2.7 (Calculus of weak differentiation)**.** *Let U be an open subset of $\mathbb{R}^n$.*

(i) (Higher-order derivatives). Assume that $u \in L^1_{\text{loc}}(U)$, and the weak derivatives $D^\alpha u$ and $D^\beta u$ exist for multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. Then if any one of the weak derivatives $D^\alpha(D^\beta u), D^\beta(D^\alpha u), D^{\alpha+\beta}u$ exists, all three weak derivatives exist and are equal.

(ii) (Leibniz product rule). Assume that $\psi \in C^\infty(U)$. If $u \in L^1_{\text{loc}}(U)$ is weakly differentiable, so is the product $u\psi$, and the weak gradient is

$$D(u\psi) = u\nabla\psi + \psi Du. \tag{2.1}$$

More generally, if the weak derivative $D^\alpha u$ exists for $\alpha \in \mathbb{N}_0^n$, then

$$D^\alpha(u\psi) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}D^\beta u\,\partial^{\alpha-\beta}\psi. \tag{2.2}$$

(iii) (Chain rule). Assume that $F \in C^1(\mathbb{R})$, and its derivative $F' \in L^\infty(\mathbb{R})$ is bounded. If $u \in L^1_{\text{loc}}(U)$ is weakly differentiable, so is the composite function $F \circ u$, and

$$D(F\circ u) = F'(u)\cdot Du.$$

Proof. (i) Using the existence of $D^\alpha u$ and the fact that $\partial^\beta\phi \in C_c^\infty(U)$ for all $\phi \in C_c^\infty(U)$, one have

$$\int_U D^\alpha u\,\partial^\beta\phi\,dm = (-1)^{|\alpha|}\int_U u\partial^{\alpha+\beta}\phi\,dm.$$

Hence $D^{\alpha+\beta}u$ exists if and only if $D^\beta(D^\alpha u)$ exists, and $D^\beta(D^\alpha u) = D^{\alpha+\beta}u$ in the weak sense. A symmetric argument holds with α and β exchanged.

(ii) For any $\phi \in C_c^\infty(U)$, the function $\psi\phi \in C_c^\infty(U)$, and

$$\int_U(D_{x_i}u)\psi\phi\,dm = -\int_U u\partial_{x_i}(\psi\phi)\,dm = -\int_U u(\partial_{x_i}\psi)\phi\,dm - \int_U u\psi\partial_{x_i}\phi\,dm.$$

By definition, we have $D_{x_i}(u\psi) = (D_{x_i}u)\psi + u\partial_{x_i}\psi$, which is the case $\alpha = e_i$ of (2.2). Now we prove the general case by induction. Suppose formula (2.2) is valid for all multi-indices $\beta < \alpha$. We choose $\alpha = \beta + e_i$

for some $|\beta| = |\alpha| - 1$ and $i \in [n]$. Then for any $\phi \in C_c^\infty(U)$, by the assumption of induction, we have

$$\int_U u\psi\partial^\alpha\phi\, dm = \int_U u\psi\partial^\beta(\partial^{e_i}\phi)\, dm = (-1)^{|\beta|}\int_U \sum_{\gamma\le\beta}\binom{\beta}{\gamma} D^\gamma u\,\partial^{\beta-\gamma}\psi\,\partial_{x_i}\phi\, dm.$$

Using the product rule, we have

$$\begin{aligned}
\int_U u\psi\partial^\alpha\phi\, dm &= (-1)^{|\beta|+1}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\int_U D^{e_i}(D^\gamma u\,\partial^{\beta-\gamma}\psi)\phi\, dm \\
&= (-1)^{|\alpha|}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\int_U \left(D^{\gamma+e_i}u\,\partial^{\alpha-\gamma-e_i}\psi + D^\gamma u\,\partial^{\alpha-\gamma}\psi\right)\phi\, dm \\
&= (-1)^{|\alpha|}\sum_{\gamma\le\beta+e_i}\int_U \left(\binom{\beta}{\gamma-e_i}D^\gamma u\,\partial^{\alpha-\gamma}\psi + \binom{\beta}{\gamma}D^\gamma u\,\partial^{\alpha-\gamma}\psi\right)\phi\, dm \\
&= (-1)^{|\alpha|}\int_U \left(\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}D^\gamma u\,\partial^{\alpha-\gamma}\psi\right)\phi\, dm.
\end{aligned}$$

(iii) Since $F' \in L^\infty(\mathbb{R})$, the function F is globally Lipschitz, and we suppose $|F(t) - F(s)| \le L|t-s|$. By Theorem 2.6, we choose a sequence $u_m \in C^\infty(U)$ such that $u_m \to u$ and $\partial_{x_i}u_m \to \partial_{x_i}u$ in $L^1_{\mathrm{loc}}(U)$. Let $v = F\circ u$, and $v_m = F\circ u_m \in C^1(U)$, with $\partial_{x_i}v_m = F'(u_m)\partial_{x_i}u_m \in C(U)$. If $V \Subset U$, then

$$\int_V |v_m - v|\, dm = \int_V |F(u_m) - F(u)|\, dm \le L\int_V |u_m - u|\, dm \to 0 \quad as \quad n\to\infty.$$

Furthermore, for the partial derivatives, we have

$$\begin{aligned}
\int_V |\partial_{x_i}v_m - F'(u)D_{x_i}u|\, dm &= \int_V |F'(u_m)\partial_{x_i}u_m - F'(u)D_{x_i}u|\, dm \\
&\le \int_V |F'(u_m)|\,|\partial^{e_i}u_m - D_{x_i}u|\ dm + \int_V |F'(u_m) - F'(u)|\,|D_{x_i}u|\ dm \\
&\le L\int_V |\partial_{x_i}u_m - D_{x_i}u|\ dm + \int_V \underbrace{|F'(u_m) - F'(u)|\,|D_{x_i}u|}_{\le 2L|D_{x_i}u|\,\in\, L^1(V)}\ dm.
\end{aligned}$$

Using the fact that $\partial_{x_i}u_m \to D_{x_i}u$ in $L^1_{\mathrm{loc}}(U)$ and the Dominated Convergence Theorem, the last display converges to zero. Since $V \Subset U$ is arbitrary, we have $v_m \to v$ and $\partial_{x_i}v_m \to F'(u)D_{x_i}u$ in $L^1_{\mathrm{loc}}(U)$. Again by Theorem 2.6, we have $D_{x_i}(F\circ u) = D_{x_i}v = F'(u)D_{x_i}u$. □

Remark. Using a similar approximation argument applied in the proof of (iii), we can show that the product rule (2.1) holds for all $\psi \in C^1(U)$ and all weakly differentiable $u \in L^1_{\mathrm{loc}}(U)$.

Proposition 2.8. *Let U be an open subset of $\mathbb{R}^n$, and $u \in L^1_{\mathrm{loc}}(U)$. If u is weakly differentiable, then both $u^+ = \max\{u, 0\}$ and $u^- = \{-u, 0\}$ are weakly differentiable, and*

$$Du^+ = \begin{cases} Du & a.e.\ on\ \{u > 0\}, \\ 0 & a.e.\ on\ \{u \le 0\}, \end{cases} \qquad Du^- = \begin{cases} 0 & a.e.\ on\ \{u > 0\}, \\ -Du & a.e.\ on\ \{u \le 0\}. \end{cases}$$

Proof. For each $\epsilon > 0$, we define $F_\epsilon \in C^1(\mathbb{R})$ as follows:

$$F_\epsilon(z) = \begin{cases} \sqrt{z^2+\epsilon^2} - \epsilon, & z > 0, \\ 0, & z \le 0. \end{cases}$$

Then $u^+ = \lim_{\epsilon\downarrow 0} F_\epsilon(u)$ in U. Also, the derivative

$$F_\epsilon'(z) = \begin{cases} \frac{z}{\sqrt{z^2+\epsilon^2}}\chi_{(0,\infty]}(z), & z > 0, \\ 0, & z \le 0. \end{cases}$$

is bounded, and $\chi_{\{u>0\}} = \lim_{\epsilon\downarrow 0} F_\epsilon'(u)$. By Proposition 2.7 (iv), for each $\varphi \in C_c^\infty(U)$,

$$\int_U F_\epsilon'(u(x))Du(x)\varphi(x)\,dx = \int_U F_\epsilon(u(x))D\varphi(x)\,dx.$$

By domanited convergence theorem, we let $\epsilon \downarrow 0$ to obtain

$$\int_U \varphi\chi_{\{u>0\}}Du\,dx = \int_U u^+ D\varphi\,dx.$$

Hence $\chi_{\{u>0\}}Du$ is the weak gradient of u^+. A similar result for u^- is obtained by considering $-u$. □

Remark. According to our result, if u is weakly differentiable in U, then $Du = 0$ a.e. on $\{u = 0\}$. More generally, $Du = 0$ a.e. on any set where u is constant. Furthermore, since $|u| = u^+ + u^-$, we know that $|u|$ is also weakly differentiable, and

$$D|u| = \begin{cases} Du & \text{a.e. on } \{u > 0\}, \\ 0 & \text{a.e. on } \{u = 0\}, \\ -Du & \text{a.e. on } \{u < 0\}. \end{cases}$$

Proposition 2.9 (Generalized product rule)**.** *Let U be an open subset of $\mathbb{R}^n$. If $u, v \in L^1_{\text{loc}}(U)$ are weakly differentiable, $uv \in L^1_{\text{loc}}(U)$ and $uDv, vDu \in L^1_{\text{loc}}(U;\mathbb{R}^n)$, then uv is weakly differentiable, and*

$$D(uv) = uDv + vDu. \tag{2.3}$$

Proof. We first assume that u, v are bounded in U. By Theorem 2.6, take a sequence of functions $u_m \in C^\infty(U)$ with $u_m \to u$ and $\nabla u_m \to Du$ in $L^1_{\text{loc}}(U)$. By the product rule (2.1), we see that

$$\int_U u_m v\,\partial_{x_i}\phi\,dx = -\int_U (v\partial_{x_i}u_m + u_m D_{x_i}v)\phi\,dx.$$

Since u, v are bounded, we let $m \to \infty$ to obtain (2.3). For the general case, we take $u_k = \max\{\min\{u,k\}, -k\}$ and $v_k = \max\{\min\{v,k\}, -k\}$, where $k \in \mathbb{N}$. Then $u_k v_k \to uv$ with $|u_k v_k| \le |uv|$, and by Proposition 2.8, $u_k Dv_k + v_k Du_k \to uDv + vDu$ with $|u_k Dv_k + v_k Du_k| \le |uDv| + |vDu|$. The result follows by applying dominated convergence theorem when $k \to \infty$. □

Proposition 2.10 (Generalilzed chain rule)**.** *Let F be a continuous function on $\mathbb{R}$ with piecewise continuous first derivative $F' \in L^\infty(\mathbb{R})$. If $u \in L^1_{\text{loc}}(U)$ is weakly differentiable, so is the composite function $F \circ u$. Furthermore, if $K \subset \mathbb{R}$ is the set of knots of F, then*

$$D(F \circ u) = \begin{cases} F'(u)\cdot Du & \text{on } u \notin K, \\ 0 & \text{on } u \in K. \end{cases}$$

Proof. By an induction argument, the proof is reduced to the case of one knot which we may take without loss of generality at the origin, i.e. $K = \{0\}$. Let $F_1, F_2 \in C^1(\mathbb{R})$ satisfy $F_1', F_2' \in L^\infty(\mathbb{R})$, with $F_1(u) = F(u)$ for $u \ge 0$, and $F_2(u) = F(u)$ for $u \le 0$. Then we have $F(u) = F_1(u^+) + F_2(-u^-)$, and the result follows from Proposition 2.7 (iv) and the Remark (i) under Proposition 2.8. □

2.3 Sobolev Spaces and Approximation

Sobolev spaces consist of functions whose weak derivatives belong to L^p. These spaces provide one of the most useful settings for the analysis of PDEs.

Definition 2.11 (Sobolev spaces)**.** Let U be an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, and $1 \le p \le \infty$. The *Sobolev space* $W^{k,p}(U)$ consists of all locally integrable functions $u : U \to \mathbb{R}$ such that for each multi-index α with $|\alpha| \le k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(U)$. We identify functions in $W^{k,p}(U)$ which agree a.e., and define the norm of $u \in W^{k,p}(U)$ to be

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_U |D^\alpha u|^p \, dm\right)^{1/p}, & 1 \le p \le \infty, \\ \max_{|\alpha|\le k} \operatorname{ess\,sup}_U |D^\alpha u|, & p = \infty. \end{cases}$$

We write $H^k(U) = W^{k,2}(U)$, where we define the inner product $\langle u, v\rangle_{H^k(U)} := \sum_{|\alpha|\le k} \int_U D^\alpha u \, D^\alpha v \, dm.$

Remark. (I) We need to check that $\|\cdot\|_{W^{k,p}(U)}$ is a norm on $W^{k,p}(U)$. Nonnegativeness and homogeneity of $\|\cdot\|_{W^{k,p}(U)}$ are clear, and the triangle inequality is also clear when $p = \infty$. Hence we only verify the triangle inequality in the case $1 \le p \le \infty$. By Minkowski's inequality,

$$\begin{aligned} \|u+v\|_{W^{k,p}(U)} &= \left(\sum_{|\alpha|\le k} \|D^\alpha u + D^\alpha v\|_{L^p(U)}^p\right)^{1/p} \le \left(\sum_{|\alpha|\le k} \left(\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)}\right)^p\right)^{1/p} \\ &\le \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(U)}^p\right)^{1/p} + \left(\sum_{|\alpha|\le k} \|D^\alpha v\|_{L^p(U)}^p\right)^{1/p} = \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}. \end{aligned}$$

(II) Corresponding to Propositiona 2.7 and 2.8, the following properties of Sobolev spaces holds:

(i) If $k \le l$, then $W^{k,p}(U) \subset W^{l,p}(U)$. If $u \in W^{k,p}(U)$, then $D^\alpha u \in W^{k-|\alpha|,p}(U)$ for all $|\alpha| \le k$.

(ii) If $u \in W^{k,p}(U)$ and $\psi \in C^\infty(U)$, then $u\psi \in W^{k,p}(U)$;

(iii) If $u \in W^{1,p}(U)$ and $F \in C^1(\mathbb{R})$, then $F \circ u, u^+, u^-, |u| \in W^{1,p}(U)$.

The Sobolev spaces have a nice structure.

Theorem 2.12. *For each $k \in \mathbb{N}$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.*

Proof. We need to show that $W^{k,p}(U)$ is complete. Let $(u_m)_{m=1}^\infty$ be a Cauchy sequence in $W^{k,p}(U)$. Then for each $|\alpha| \le k$, $(D^\alpha u_m)_{m=1}^\infty$ is a Cauchy sequence in $L^p(U)$. By completeness of $L^p(U)$, there exists $u^{(\alpha)} \in L^p(U)$ such that $D^\alpha u_m \to u^{(\alpha)}$ in $L^p(U)$ for each $|\alpha| \le k$, and in particular $u_m \to u$ in $L^p(U)$ when $\alpha = 0$.

Clearly, if we can show that $u \in W^{k,p}(U)$ and $D^\alpha u = u^{(\alpha)}$ for all $|\alpha| \le k$, the result follows. To this end, we let $q = \frac{p}{p-1}$ be the Hölder conjugate, and fix any $\phi \in C_c^\infty(U)$. By Hölder's inequality,

$$\left|\int_U (u_m - u)\partial^\alpha \phi \, dx\right| \le \|u_m - u\|_{L^p(U)} \|\partial\phi\|_{L^q(U)} \to 0, \quad \textit{and} \tag{2.4}$$

$$\left|\int_U (D^\alpha u_m - u^{(\alpha)})\phi \, dx\right| \le \|D^\alpha u_m - u^{(\alpha)}\|_{L^p(U)} \|\phi\|_{L^q(U)} \to 0. \tag{2.5}$$

These two limits imply the interchangeability of the limit and the integral:

$$\int_U u\partial^\alpha \phi \, dx = \lim_{m\to\infty} \int_U u_m \partial^\alpha \phi \, dx = (-1)^{|\alpha|} \lim_{m\to\infty} \int_U D^\alpha u_m \phi \, dx = (-1)^{|\alpha|} \int_U u^{(\alpha)} \phi \, dx.$$

Hence our assertion is valid. Since $D^\alpha u_m \to D^\alpha u$ in $L^p(U)$ for all $|\alpha| \le k$, we have $u_m \to u$ in $W^{k,p}(U)$. $\square$

Definition 2.13 (Local Sobolev spaces)**.** Let U be an open subset of $\mathbb{R}^n$, $k \in \mathbb{N}$, and $1 \le p \le \infty$. The local Sobolev space $W^{k,p}_{\mathrm{loc}}(U)$ consists of all locally integrable functions $u : U \to \mathbb{R}$ whose restriction to any pre-compact $V \Subset U$ lies in $W^{k,p}(V)$, i.e.

$$W^{k,p}_{\mathrm{loc}}(U) = \left\{ u \in L^1_{\mathrm{loc}}(U) : \forall V \Subset U,\ u|_V \in W^{k,p}(V) \right\}.$$

We say a sequence of functions $u_m \in W^{k,p}_{\mathrm{loc}}(U)$ converges to u in $W^{k,p}_{\mathrm{loc}}(U)$ if $\|u_m - u\|_{W^{k,p}(V)} \to 0$ as $m \to \infty$ for all pre-compact $V \Subset U$.

Remark. To summarize, for $k \in N$ and $1 \le p \le \infty$, there are the, in general strict, inclusions

$$\begin{array}{ccccc} L^p(U) & \subset & L^p_{\mathrm{loc}}(U) & \subset & L^1_{\mathrm{loc}}(U) \\ \cup & & \cup & & \cup \\ W^{k,p}(U) & \subset & W^{k,p}_{\mathrm{loc}}(U) & \subset & W^{k,1}_{\mathrm{loc}}(U) \end{array}$$

Next we are going to discuss approximation of Sobolev functions.

Theorem 2.14 (Local approximation by smooth functions)**.** *Assume $1 \le p < \infty$. For each $u \in W^{k,p}(U)$, the function $u^\epsilon = \eta_\epsilon * \overline{u}^{(\epsilon)} \in C^\infty(U)$ for each $\epsilon > 0$, and $u^\epsilon \to u$ in $W^{k,p}_{\mathrm{loc}}(U)$ as $\epsilon \to 0$.*

Proof. According to Proposition 1.8 and the Remark under Lemma 2.5, $u^\epsilon \to u$ and $D^\alpha u^\epsilon \to D^\alpha u$ in $L^p(V)$ as $\epsilon \to 0$ for all $|\alpha| \le k$ and all pre-compact $V \Subset U$. Then

$$\|u^\epsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha| \le k} \|D^\alpha u^\epsilon - D^\alpha u\|^p_{L^p(V)} \to 0 \quad as \quad \epsilon \to 0. \tag{2.6}$$

Hence $u^\epsilon \to u$ in $W^{k,p}_{\mathrm{loc}}(U)$ as $\epsilon \to 0$. ☐

Remark. If $U = \mathbb{R}^n$, the convergence (2.6) remains valid by Proposition 1.5 when we replace V by $\mathbb{R}^n$. Consequently, $C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $1 \le p < \infty$. Now we assume $u \in C^\infty(\mathbb{R}^n) \cap W^{k,p}(\mathbb{R}^n)$, and choose $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| \ge 2$. Let $\phi_R = \phi(x/R)$. Then $u^R := \phi_R u \in C^\infty_c(\mathbb{R}^n)$, and by Leibniz rule, we have

$$D^\alpha u^R = \phi_R D^\alpha u + \frac{1}{R} h_R \to D^\alpha u, \quad as \quad R \to \infty,$$

where h_R is bounded in L^p uniformly in R. Hence $u^R \to u$ in $W^{k,p}(\mathbb{R}^n)$ as $R \to \infty$. Therefore, the space $C^\infty_c(\mathbb{R}^n)$ is dense in $W^{k,p}(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $1 \le p < \infty$.

We denote by $W^{k,p}_0(U)$ the closure of $C^\infty_c(U)$ in $W^{k,p}(U)$:

$$W^{k,p}_0(U) := \overline{C^\infty_c(U)}^{\|\cdot\|_{W^{k,p}(U)}}$$

For the case $U = \mathbb{R}^n$, we have the result $W^{k,p}_0(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$. However, we do not have a similar global approximation conclusion for general $U \subset \mathbb{R}^n$.

Theorem 2.15 (Global approximation by smooth functions on bounded domains)**.** *Assume that $U \subset \mathbb{R}^n$ is open and bounded, and $1 \le p < \infty$. Then for each $u \in W^{k,p}(U)$, there exists a sequence of functions $u_m \in C^\infty(U) \cap W^{k,p}(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{k,p}(U)$ as $m \to \infty$.*

Proof. We write $U_r = \{x \in U : d(x, \partial U) > 1/r\}$, and $V_r := U_{r+3} \backslash \overline{U}_{r+1}$, where $r = 1, 2, \cdots$. Take any open $V_0 \Subset U_4$ such that $U = \bigcup_{r=0}^\infty V_r$, and choose a smooth partition of unity $\phi_r : U \to [0,1]$ subordinate to $(V_r)_{r=0}^\infty$:

$$\phi_r \in C^\infty_c(V_r), \quad \sum_{r=0}^\infty \phi_r = 1 \ on\ U.$$

Then for any $u \in W^{k,p}(U)$, we have $\phi_r u \in W^{k,p}(U)$ and $\mathrm{supp}(\phi_r u) \Subset V_r$. Now fix $\delta > 0$, and choose $\epsilon_r > 0$ so small that $u^r = \eta * (\phi_r u)$ satisfies

$$\|u^r - \phi_r u\|_{W^{k,p}(U)} \leq \frac{\delta}{2^{r+1}},\ r = 0, 1, 2, \cdots; \quad \mathrm{supp}\, u^r \subset U_{r+4} \backslash \overline{U}_r,\ r = 1, 2, \cdots.$$

Let $v = \sum_{r=0}^{\infty} u^r$. Then $v \in C^\infty(U)$, since for each open set $V \Subset U$ there are at most finitely many nonzero terms in the sum. Furthermore,

$$\|v - u\|_{W^{k,p}(V)} \leq \sum_{r=0}^{\infty} \|u^r - \phi_r u\|_{W^{k,p}(U)} \leq \delta \sum_{r=1}^{\infty} \frac{1}{2^{r+1}} = \delta.$$

Taking the supremum over open sets $V \Subset U$, we conclude that $\|v - u\|_{W^{k,p}(U)} \leq \delta$. □

Now we discuss the approximation of Sobolev functions even up to the boundary of domain U. To prepare, we introduce some regularity conditions on boundaries.

Definition 2.16 (Regularity of boundaries)**.** For a pre-compact $U \Subset \mathbb{R}^n$, its boundary ∂U is said to be *Lipschitz*, if for each $x^0 \in \partial U$, there exists a radius $r > 0$ and a Lipschitz continuous map $\gamma : \Omega \to \mathbb{R}$, defined on an open set $\Omega \subset \mathbb{R}^{n-1}$ with Lipschitz constant, say L_γ, such that, after possibly relabeling and reorienting some coordinate axes, (i) the part of the boundary ∂U inside the closed ball $B(x^0, r)$ is the graph of γ, and (ii) the part of U inside the closed ball $B(x^0, r)$ is of the simple form

$$U \cap B(x^0, r) = \left\{x \in B(x^0, r) : x_n > \gamma(x_1, \cdots, x_n)\right\}.$$

In addition, for any $k \in \mathbb{N} \cup \{\infty\}$, ∂U is said to be C^k if $\gamma \in C^k(\Omega)$.

Remark. By compactness of ∂U, we can choose finitely many tuples $(x_1^0, r_1, \gamma_1), \cdots, (x_N^0, r_N, \gamma_N)$ such that the open balls $B^0(x_1^0, r_1), \cdots .B^0(x_N^0, r_N)$ cover ∂U. Consequently, the Lipschitz maps γ we choose are *uniformly Lipschitz.* In other words, for all $x^0 \in \partial U$, the map γ we choose to describe the local geometry of ∂U has Lipschitz constant smaller than $\gamma := \max_{1 \leq j \leq N} \gamma_j$.

In a domain U whose boundary ∂U is Lipschitz, we can approximate a Sobolev function using functions smooth up to the boundary, i.e. the functions in $C^\infty(\overline{U})$.

Theorem 2.17 (Global approximation by functions smooth up to the boundary of Lipschitz domains)**.** *Assume that $U \subset \mathbb{R}^n$ is open and bounded, ∂U is Lipschitz, and $1 \leq p < \infty$. Then for each $u \in W^{k,p}(U)$, there exists a sequence of functions $u_m \in C^\infty(\overline{U})$ such that $u_m \to u$ in $W^{k,p}(U)$ as $m \to \infty$.*

Proof. Step I. In this step, we construct a space for mollification within U. Given $x^0 \in \partial U$, we pick a radius $r > 0$ and a Lipschitz map γ whose graph is part of ∂U inside $B(x^0, r)$. Define the closed horizontal double cone $\widetilde{C}_0$ and open upward cone C_0:

$$\widetilde{C}_0 = \{(x', x_n) \in \mathbb{R}^n : |x_n| \leq L|x'|\}, \quad C_0 = \{(x', x_n) \in \mathbb{R}^n : x_n > L|x'|\}.$$

Then for any $y \in \partial U$, the translated horizontal double cone $\widetilde{C}_y = y + \widetilde{C}_0$ contains $\partial U \cap B(y, r(y))$, and the translated open upward cone $C_y = y + C_0$ lies in U within some radius $r(y)$ from y.

Let $V = U \cap B^0(x^0, r/2)$. For any $x \in V$, define the upward shifted point

$$x^\epsilon := x + \epsilon\lambda e_n, \quad x \in V,\ \epsilon > 0,$$

where $\lambda > \sqrt{1 + L^2}$ is so large that the ball $B(x^\epsilon, \epsilon)$ lies in the upward cone $C_{\tilde{x}}$ for all $0 < \epsilon < 1$, where $\tilde{x} \in \partial U \cap B(x_0, r/2)$ shares the same horizontal coordinates with x. Moreover, for all $\epsilon > 0$ sufficiently small, the family $B(x^\epsilon, \epsilon)$ is located near x, hence in the open neighborhood $W := U \cap B^0(x^0, r)$ for all $x \in V$.

Now we define $u_\epsilon(x) = u(x^\epsilon)$ for all $x \in V$, which is the function u translated a distance $\lambda\epsilon$ in the e_n direction. Write $v^\epsilon = \eta_\epsilon * u_\epsilon$. Then v^ϵ is not only defined on V, because for any $\tilde{x} \in \partial U \cap B(x_0, r/2)$,

$$v^\epsilon(\tilde{x}) = \int_{B(\tilde{x},\epsilon)} \eta_\epsilon(\tilde{x} - y) u_\epsilon(y)\, dy = \int_{B(\tilde{x},\epsilon)} \eta_\epsilon(\tilde{x} - y) u(\underbrace{y + \epsilon\lambda e_n}_{\in B(\tilde{x}+\epsilon\lambda e_n, \epsilon)})\, dy.$$

Since $B(\tilde{x} + \epsilon\lambda e_n, \epsilon) \subset C_{\tilde{x}}$, $v^\epsilon(\tilde{x})$ is well-defined. Consequently, v^ϵ is also defined on a sufficiently small neighborhood of $\tilde{x} \in \partial V \cap \partial U$, and $v^\epsilon \in C^\infty(\overline{V})$.

Step II. We prove that $v^\epsilon \to u$ in $W^{k,p}(V)$. To this end, we take any multi-index $|\alpha| \le k$. Then

$$\begin{aligned}
\|\partial^\alpha v^\epsilon - D^\alpha u\|_{L^p(V)} &\le \|\partial^\alpha v^\epsilon - D^\alpha u_\epsilon\|_{L^p(V)} + \|D^\alpha u_\epsilon - D^\alpha u\|_{L^p(V)} \\
&= \|\eta_\epsilon * (D^\alpha u_\epsilon) - D^\alpha u_\epsilon\|_{L^p(V)} + \|D^\alpha u_\epsilon - D^\alpha u\|_{L^p(V)} \\
&\le \|\eta_\epsilon * (D^\alpha u) - D^\alpha u\|_{L^p(\mathbb{R}^n)} + \|D^\alpha u_\epsilon - D^\alpha u\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

The first term vanishes as $\epsilon \to 0$ by Proposition 1.6, and the second term also vanishes by continuity of translation operator in L^p-norm.

Step III. We finally prove the global result via partition of unity. Pick $\delta > 0$. By compactness of ∂U, there exist finitely many points $x_i^0 \in \partial U$, radii $r_i > 0$, corresponding sets $V_i = U \cap B^0(x_i^0, \frac{r_i}{2})$ and functions $v^i \in C^\infty(V_i)$, where $i = 1, \cdots, N$ such that the open balls $B^0(x_i^0, \frac{r_i}{2})$ form a cover of ∂U, and (by Step II)

$$\|v^i - u\|_{W^{k,p}(V_i)} < \delta.$$

Choose $V_0 \Subset U$ such that $(V_i)_{i=0}^N$ is an open cover U, and $v^0 \in C^\infty(\overline{V}_0)$ such that $\|v^0 - u\|_{W^{k,p}(V_0)} < \delta$ by Theorem 2.14. By taking a smooth partition of unity $(\phi_i)_{i=0}^N$ subordinate to the open cover, we construct a smooth function $v = \sum_{i=0}^N \phi_i v_i \in C^\infty(\overline{U})$. Furthermore, for each $|\alpha| \le k$, one have

$$\begin{aligned}
\|D^\alpha v - D^\alpha u\|_{L^p(U)} &\le \sum_{i=1}^N \|D^\alpha(\phi_i v_i) - D^\alpha(\phi_i u)\|_{L^p(V_i)} \\
&\le \sum_{i=1}^N \left\| \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \phi_i (D^{\alpha-\beta} v_i - D^{\alpha-\beta} u) \right\|_{L^p(V_i)} \\
&\le C \sum_{i=1}^N \|v_i - u\|_{W^{k,p}(U)} \le C(N+1)\delta
\end{aligned}$$

for some constant $C = C(k,p) > 0$. Since $\delta > 0$ can be arbitrarily small, the proof is completed. □

2.4 Absolute Continuity on Lines

In this section, we discuss the relation between the weak partial derivatives and the classical partial derivatives. Throughout this discussion, the absolute continuity of functions restricted to line segments plays an important role. Keep in mind that we identify functions that agree a.e..

Theorem 2.18 (ACL characterization). *Let $1 \le p \le \infty$ and $u \in L^p(U)$. Then $u \in W^{1,p}(U)$ if and only if u has a representative $\overline{u}$ that has the ACL property, i.e. $\overline{u}$ is $\underline{a}$bsolutely $\underline{c}$ontinuous on almost all $\underline{l}$ine segments in U parallel to the coordinate axes and whose (classical) partial derivatives exist a.e. and belong to $L^p(U)$. Moreover, the (classical) partial derivatives of $\overline{u}$ agree a.e. with the weak derivatives of u.*

Proof. Step I. We first suppose that $u \in W^{1,p}(U)$, and find its representative $\overline{u}$ having the desired property.

Case I: $1 \le p < \infty$. Write $x \in I$ as $x = (x_{-i}, x_i)$, where

$$x_{-i} \in U_i := \left\{ t_{-i}\mathbb{R}^{n-1} : \{(t_{-i}, t_i) : t_i \in \mathbb{R}\} \cap U \neq \emptyset \right\}, \quad \textit{and} \quad x_i \in U_{x_{-i}} := \{t_i \in \mathbb{R} : (x_{-i}, t_i) \in U\}$$

By Theorem 2.14, the mollifiers u^ϵ converges to u in $W^{k,p}(V)$ for any $V \Subset U$. By Fubini's theorem,

$$\lim_{\epsilon \to 0} \int_{U_i} \int_{V_{x_{-i}}} \sum_{|\alpha| \le 1} |D^\alpha u^\epsilon(x_{-i}, x_i) - D^\alpha u(x_{-i}, x_i)|^p dx_i \, dx_{-i} = 0.$$

Consequently, we can find a subsequence $\epsilon_l \to 0$ such that

$$\lim_{l \to \infty} \int_{V_{x_{-i}}} \sum_{|\alpha| \le 1} |D^\alpha u^{\epsilon_l}(x_{-i}, x_i) - D^\alpha u(x_{-i}, x_i)|^p dx_i = 0 \quad \textit{for a.e. } x_{-i} \in U_i. \tag{2.7}$$

Denote $u_l = u^{\epsilon_l}$, and let $\overline{u} = \lim_{l \to \infty} u_l$. By Proposition 1.8, $\overline{u}$ agrees with u except on a Lebesgue null set $E \subset U$. Again by Fubini's theorem,

$$\int_{U_i} \int_{U_{x_{-i}}} \sum_{|\alpha|=1} |D^\alpha u(x_{-i}, x_i)|^p \, dx_i \, dx_{-i} < \infty, \qquad \int_{U_i} \mathcal{L}^1(\{x_i \in U_{x_{-i}} : (x_{-i}, x_i) \in E\}) \, dx_{-i} = 0.$$

Correspondingly, we may find a set $N_i \subset U_i$ with $\mathcal{L}^{n-1}(N_i) = 0$ such that for all $x_{-i} \in U_i \backslash N_i$,

$$\int_{U_{x_{-i}}} \sum_{|\alpha|=1} |D^\alpha u(x_{-i}, x_i)|^p \, dx_i < \infty, \quad \mathcal{L}^1(\{x_i \in U_{x_{-i}} : (x_{-i}, x_i) \in E\}) = 0.$$

Fix any such x_{-i}, and let $I \subset U_{x_{-i}}$ be a maximal open interval. Fix $t_0 \in I$ with $(x_{-i}, t_0) \in U \backslash E$, and let $t \in I$. Then there exists an open set $V \Subset U$ containing both (x_{-i}, t_0) and (x_{-i}, t). Since $u_l \in C^\infty(V)$, by fundamental theorem of calculus, one have

$$u_l(x_{-i}, t) = u_l(x_{-i}, t_0) + \int_{t_0}^t \partial_{x_i} u_l(x_{-i}, s) \, ds.$$

Since $(x_{-i}, t_0) \in U \backslash E$, we have $u_l(x_{-i}, t_0) \to \overline{u}(x_{-i}, t_0)$. Moreover, by (2.7),

$$\lim_{l \to \infty} \int_{t_0}^t |\partial_{x_i} u_l(x_{-i}, s) - D_{x_i} u(x_{-i}, s)| \, ds = 0.$$

Therefore, once $(x_{-i}, t_0) \in U \backslash E$, which holds for a.e. $t \in I$, we have

$$\overline{u}(x_{-i}, t) = \overline{u}(x_{-i}, t_0) + \int_{t_0}^t \partial_{x_i} u(x_{-i}, s) \, ds.$$

It is seen that the function $\overline{u}(x_{-i},\cdot)$ is absolutely continuous in I, and $\partial_{x_i}\overline{u} = D_{x_i}u$ for a.e. $t \in I$.

CASE II: $p = \infty$. We first consider an open ball $B \Subset U$, and prove that u is Lipschitz in B. Since $u \in W^{1,\infty}(U)$, there exists $M > 0$ such that $\operatorname{ess\,sup}_U |Du| \le M$. Then for all $\epsilon > 0$ small enough,

$$u^\epsilon(x) = (\eta_\epsilon * u)(x) \quad \textit{and} \quad \partial_{x_i}u^\epsilon(x) = (\eta_\epsilon * D_{x_i}u)(x),\ i = 1,\cdots,n, \quad \forall x \in B.$$

Hence $\|u^\epsilon\|_{L^\infty(B)} \le \|u\|_{L^\infty(B)}$, and $\sup_B |\nabla u^\epsilon| \le \operatorname{ess\,sup}_B \|Du\|_\infty \le M$. This implies that the family (u^ϵ) is uniformly bounded and equicontinuous:

$$|u^\epsilon(x) - u^\epsilon(y)| \le M|x-y|.$$

By Arzelà-Ascoli theorem, we may find a subsequence $\epsilon_l \to 0$ such that $u_l := u^{\epsilon_l}$ converges uniformly to a function $\overline{u} : B \to \mathbb{R}$ as $l \to \infty$, and $|\overline{u}(x) - \overline{u}(y)| \le M|x-y|$. Note $u = \overline{u}$ a.e. in B.

By covering U with countably many balls and applying the standard diagonal trick, we can extend u to a continuous function $\overline{u} : U \to \mathbb{R}$ such that $u = \overline{u}$ a.e..

Now we prove that $\overline{u}$ is Lipschitz on all segments I in U. If I falls in a ball, the result is clear. Otherwise, by compactness of I, we can find finitely many balls B_i covering I and points $x_0, x_1, \cdots, x_N \in U$ such that the segment $I = \{tx_0 + (1-t)x_N : t \in [0,1]\}$ consists of N subsegments $I_i = \{tx_{i-1} + (1-t)x_i : t \in [0,1]\} \subset B_i$, where $i = 1,\cdots,N$. For any $x, y \in I$, with $x_{j+1}, x_{j+2}, \cdots, x_k \in \{tx + (1-t)y : t \in [0,1]\}$, we have

$$\begin{aligned} |u(x) - u(y)| &\le |u(x) - u(x_j)| + |u(x_{j+1}) - u(x_j)| + \cdots + |u(x_k) - u(x_{k-1})| + |u(y) - u(x_k)| \\ &\le M|x - x_j| + M|x_j - x_{j-1}| + \cdots + M|x_k - x_{k-1}| + M|y - x_k| = M|x-y|. \end{aligned}$$

Hence $\overline{u}$ is Lipschitz on I. If I is parallel to any coordinate axis, the partial derivative of $\overline{u}$ with respect to the corresponding variable is bounded by M. Hence $\partial_{x_i}\overline{u} \in L^\infty(U)$.

Step II. Conversely, let $\overline{u}$ be the representative of u having the desired property. Fix $i = 1,\cdots,n$ and let $x_{-i} \in U_i$ be such that $\overline{u}(x_{-i},\cdot)$ is absolutely continuous on every connected component of the open set $U_{x_{-i}}$. Then for every $\phi \in C_c^\infty(U)$, $\overline{u}(x_{-i},\cdot)\phi(x_{-i},\cdot)$ is absolutely continuous. By the integration by parts formula,

$$\int_{U_{x_{-i}}} \overline{u}(x_{-i},t)\partial_{x_i}\phi(x_{-i},t)\,dt = -\int_{U_{x_{-i}}} \partial_{x_i}\overline{u}(x_{-i},t)\phi(x_{-i},t)\,dt,$$

which holds for a.e. $x_{-i} \in U_i$. Integrating over U_i and using Fubini's theorem yields

$$\int_U \overline{u}(x)\partial_{x_i}\phi(x)\,dx = \int_U \partial_{x_i}\overline{u}(x)\phi(x)\,dx.$$

Therefore, $D^{e_i}\overline{u} = \partial^{e_i}\overline{u} \in L^p(U)$ for all $i = 1,\cdots,n$, and $u \in W^{1,p}(U)$. □

Remark. In the case $W^{1,\infty}(U)$, we did not require I to be coordinate-aligned, and the Lipschitz property holds on all line segments. We next introduce a very useful characterization of space $W^{1,\infty}(U)$.

Theorem 2.19. *Let $U \subset \mathbb{R}^n$ be a convex set. Then $C^{0,1}(\overline{U}) = W^{1,\infty}(U)$.*

Proof. Step I. Let $u \in C^{0,1}(\overline{U})$. Then u is Lipschitz on every segment parallel to coordinates axis, with partial derivatives bounded by $[u]_{C^{0,1}(\overline{U})}$. This implies $u \in W^{1,\infty}(U)$.

Step II. Conversely, let $u \in W^{1,\infty}(U)$. According to our construction of $\overline{u}$ in the Step I in the proof of Theorem 2.18, u admits a representative $\overline{u}$ that is Lipschitz on all line segments in U with Lipschitz constant $M \ge \operatorname{ess\,sup}_U |Du|$. Since U is convex, the line segment connecting any two points $x, y \in U$ lies in U, and the global Lipschitzness follows. Noticing that $u \in L^\infty(U)$, we have $u \in C^{0,1}(\overline{U})$. □

3 Extensions and Traces

3.1 Extensions

In this section, we discuss the extension of functions in the Sobolev space. Whereas in the realm of L^p spaces extending an L^p function on a domain $U \subset \mathbb{R}^n$ to all $\mathbb{R}^n$ within L^p is trivial, just extend naturally by zero. This does not work for Sobolev spaces, already not for those of first order $W^{1,p}$. A key point is to jump singularities across ∂ that obstruct existence of weak derivatives. We let $1 \le p \le \infty$ throughout this section.

Theorem 3.1 (Extension)**.** *Assume that $U \Subset \mathbb{R}^n$ is bounded and ∂U is Lipschitz. Then for any bounded open set V that contains the closure of U, in symbols $U \Subset V \Subset \mathbb{R}^n$, there is a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(V) \hookrightarrow W^{1,p}(\mathbb{R}^n), \quad u \mapsto Eu = \overline{u},$$

such that (i) $\overline{u}|_U = u$ a.e.; (ii) $\overline{u}$ is compactly supported in V; and (iii)

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} = \|\overline{u}\|_{W^{1,p}(V)} \le c\|u\|_{W^{1,p}(U)}, \tag{3.1}$$

where $c > 0$ is a constant depending on n, p, U and V.

Remark. The function $Eu = \overline{u}$ is called an *extension* of u on $\mathbb{R}^n$.

Proof. Step I. In this step, we derive the extension operator in the half ball model. Let $B \subset \mathbb{R}^n$ be the open ball with center x^0 lying in the hyperplane $\{x_n = 0\}$ and of radius r. Define

$$B_+ := B \cap \{x_n > 0\}, \quad B_- := B \cap \{x_n < 0\}.$$

We prove that there exists a linear map

$$E_0 : W^{1,p}(B_+) \to W^{1,p}(B), \quad u \mapsto E_0 u = \overline{u}$$

such that $\overline{u}|_{B+} = u$, and

$$\|\overline{u}\|_{W^{1,p}(B)} \le 16\|u\|_{W^{1,p}(B_+)}. \tag{3.2}$$

Case I: $1 \le p < \infty$. Without loss of generality, we suppose $u \in C^1(\overline{B}_+)$. By Theorem 2.17, the first two spaces in the inclusion $C^\infty(\overline{B}_+) \subset C^1(\overline{B}_+) \subset W^{1,p}(B_+)$ are both dense in $W^{1,p}(B_+)$. Therefore, if we can construct a linear operator $E_0 : C^1(\overline{B}_+) \to C^1(\overline{B})$ satisfying (3.2), then we can extend it to $E_0 : W^{1,p}(B_+) \to W^{1,p}(B)$ by a density argument and completeness of $W^{1,p}(B)$. To this end, we define

$$\overline{u}(x) = \begin{cases} u(x), & x \in \overline{B}_+, \\ -3u(x', -x_n) + 4u(x', -\frac{x_n}{2}), & x = (x', x_n) \in \overline{B}_-. \end{cases}$$

We claim that $\overline{u} \in C^1(\overline{B})$. To check this, we write $u^+ = \overline{u}|_{\overline{B}_+}$ and $u^- = \overline{u}|_{\overline{B}_-}$. Clearly, we have $u^+ = u^-$ on $B \cap \{x_n = 0\}$. Furthermore,

$$\begin{aligned} \partial_{x_i} u^-(x', x_n) &= -3\partial_{x_i} u(x', -x_n) + 4\partial_{x_i} u\left(x', -\frac{x_n}{2}\right), \quad i = 1, \cdots, n-1, \\ \partial_{x_n} u^-(x', x_n) &= 3\partial_{x_n} u(x', -x_n) - 2\partial_{x_n} u\left(x', -\frac{x_n}{2}\right). \end{aligned}$$

Hence we have $\partial^\alpha u^+ = \partial^\alpha u^-$ along $B \cap \{x_n = 0\}$ for all $|\alpha| \le 1$, and $\overline{u} \in C^1(\overline{B})$.

Now we derive the estimate (3.2). By Jensen's inequality,

$$|u^-(x',x_n)|^p \le 2^{p-1}\left(|3u(x',-x_n)|^p + \left|4u\left(x',-\frac{x_n}{2}\right)\right|^p\right) \le 2^{3p-1}\left(|u(x',-x_n)|^p + \left|u\left(x',-\frac{x_n}{2}\right)\right|^p\right)$$

Integrate on both sides of the last display, and change the variable x_n:

$$\|u^-\|^p_{L^p(B_-)} \le 2^{3p-1}\|u\|^p_{L^p(B_+)} + 2^{3p}\|u\|^p_{L^p(B_+)} \le 2^{3p+1}\|u\|^p_{L^p(B_+)}.$$

Similarly, we have $\|\partial_{x_i}u^-\|^p_{L^p(B^-)} \le 2^{3p+1}\|\partial_{x_i}u\|^p_{L^p(B^+)}$ for all $i=1,\cdots,n$. Henceforth,

$$\|\overline{u}\|^p_{W^{1,p}(B)} = \sum_{|\alpha|\le 1}\|\partial^\alpha \overline{u}\|^p_{L^p(B)} = \sum_{|\alpha|\le 1}\left(\|\partial^\alpha u^+\|^p_{L^p(B_+)} + \|\partial^\alpha u^-\|^p_{L^p(B_-)}\right) \le 2^{4p}\|u\|^p_{W^{1,p}(B_+)}.$$

Case II: $p=\infty$. By Theorem 2.19, we have $C^{0,1}=W^{1,\infty}$ for both B_+ and B. We then consider the map E_0 given by simple horizontal reflection:

$$E_0 : C^{0,1}(B_+) \to C^{0,1}(B), \quad u \mapsto \overline{u} : B \ni (x',x_n) \mapsto u(x',|x_n|).$$

Then $\overline{u}$ is indeed Lipschitz with the same Lipschitz constant as u, and

$$\|\overline{u}\|_{W^{1,\infty}(B)} = \max_{|\alpha|\le k} \operatorname{ess\,sup}_B |D^\alpha \overline{u}| = \max_{|\alpha|\le k} \operatorname{ess\,sup}_{B_+} |D^\alpha u| = \|u\|_{W^{1,\infty}(B_+)},$$

Step II. In this step we extend u near $x_0 \in \partial U$. If ∂U is not flat near x^0, we can find a Lipschitz map $\gamma : \mathbb{R}^{n-1} \supset \Omega \to \mathbb{R}$ with Lipschitz constant M whose graph coincides the part of ∂U within a small ball $B(x^0,r)$. Consider the neighborhoods $X = \Omega\times\mathbb{R}$ of $x^0=(x^0_{-n},x^0_n)$ and $Y=\Omega\times\mathbb{R}$ of $y^0=(x^0_{-n},0)$. Define

$$\begin{aligned}
\Phi &: X \to Y, \quad x \mapsto \Phi(x) := (x_1,\cdots,x_{n-1},x_n-\gamma(x_1,\cdots,x_{n-1})),\\
\Psi &: Y \to X, \quad y \mapsto \Psi(y) := (y_1,\cdots,y_{n-1},y_n+\gamma(y_1,\cdots,y_{n-1})).
\end{aligned}$$

Then $\Phi=\Psi^{-1}$ is a bi-Lipschitz map, since

$$|\Phi(x)-\Phi(z)| \le \sqrt{2(1+M^2)}|x-z| \quad \textit{and} \quad |\Psi(y)-\Psi(z)| \le \sqrt{2(1+M^2)}|y-z|.$$

By definition, Φ flattens ∂U near x^0. By Rademacher's Theorem, the graph map γ is differentiable for a.e. $x_{-n}\in\Omega$. Hence the linearizations of Φ and Ψ exist pointwise a.e. and, furthermore, the Jacobian is triangular with diagonal elements 1. Thus $\det D\Phi = 1 = \det D\Psi$ pointwise a.e..

Now we derive the local extension of $u\in W^{k,p}(U)$ near $x^0\in\partial D$. Pick a small ball B centered at $y^0=\Phi(x^0)$ and contained in the open neighborhood $\Phi(B^0(x^0,r))$ of y^0. Let B_+ be the upper open half ball of B, and consider the restriction of u to the open set $V=\Psi(B_+)$. Then $u\in W^{1,p}(V)$.

Next pull back $u : V\to\mathbb{R}$ to the y coordinates to obtain the function $v := u\circ\psi : B_+\to\mathbb{R}$ which lies in $W^{1,p}(B_+)$ by Proposition, and $\|v\|_{W^{1,p}(B_+)} = \|u\|_{W^{1,p}(V)}$. Then we employ the extension operator constructed in Step I to pick an extension $\overline{v}=E_0v$ of $v=u\circ\psi$ from the upper half ball B_+ to the whole ball B. The extension of u from $V=\Psi(B_+)$ to $A=\Psi(B)$ is defined by

$$\overline{u} = \overline{v}\circ\Phi \in W^{1,p}(A), \quad \|\overline{u}\|_{W^{1,p}(A)} = \|\overline{v}\|_{W^{1,p}(B)}.$$

According to estimate (3.2), we have

$$\|\overline{u}\|_{W^{1,p}(A)} = \|\overline{v}\|_{W^{1,p}(B)} \le 16\|v\|_{W^{1,p}(B^+)} = 16\|u\|_{W^{1,p}(V)}. \tag{3.3}$$

Step III. In this step, we extend u globally via a finite partition of unity. By Step II and compactness of ∂U, there exist finitely many $x_i^0 \in \partial U$ and local extensions $\overline{u}_i = \overline{v}^i \circ \Phi : A_i \to \mathbb{R}$ covering ∂U, where $i = 1, \cdots, N$. Now we pick $A_0 \Subset U$ such that $U \Subset A := \bigcup_{i=0}^N A_i \Subset \mathbb{R}^n$, and pick a smooth partition of unity $(\phi_i)_{i=0}^N$ subordinate to the open cover $(A_i)_{i=0}^N$ of U. Extend U to A by $\overline{u} = \sum_{i=0}^N \phi_i \overline{u}_i \in W^{1,p}(A)$. We then have the following estimate of $\|\overline{u}\|_{W^{1,p}(A)}$:

$$\begin{aligned}
\|\overline{u}\|_{W^{k,p}(A)} \le \sum_{i=0}^N \|\phi_i \overline{u}_i\|_{W^{1,p}(A_i)} &\le \sum_{i=0}^N 2n^{1/p} \|\phi_i\|_{W^{1,\infty}(A_i)} \|\overline{u}_i\|_{W^{k,p}(A_i)} && \text{(By product rule)} \\
&\le 2n^{1/p} \max_{1 \le i \le N} \|\phi_i\|_{W^{1,\infty}(A_i)} \sum_{i=0}^N \|\overline{u}_i\|_{W^{1,p}(A_i)} \\
&\le \underbrace{32n^{1/p}(1+N) \max_{1 \le i \le N} \|\phi_i\|_{W^{1,\infty}(A_i)}}_{=:c} \|u\|_{W^{1,p}(U)}, && \text{(By estimate (3.3))}
\end{aligned}$$

where we use $1/p = 0$ when $p = \infty$. Then c is a constant depending only on n, p and U. Furthermore, the linearity of the mapping $u \mapsto \overline{u}$ follows from E_0 in Step I.

Step IV. Given $u \in W^{1,p}(U)$ and $U \Subset V \Subset \mathbb{R}^n$, we have $U \Subset (V \cap A) \Subset \mathbb{R}^n$. We then pick up a cutoff function $\chi \in C_c^\infty(V \cap A)$ with $0 \le \chi \le 1$ and $\chi \equiv 1$ on U. Then $\chi \overline{u} \in W^{1,p}(V)$, where $\overline{u}$ constructed in Step III is restricted to V. Furthermore, we have the following estimate for $\|\chi \overline{u}\|_{W^{1,p}(V)}$:

$$\|\chi \overline{u}\|_{W^{1,p}(V)} = \|\chi \overline{u}\|_{W^{1,p}(V \cap A)} \le \|\chi \overline{u}\|_{W^{1,p}(A)} \le 2n^{1/p} \|\chi\|_{W^{1,\infty}(A)} \|\overline{u}\|_{W^{k,p}(A)} \le 2cn^{1/p} \|u\|_{W^{1,p}(U)}.$$

This completes the proof. □

Remark. (i) If $1 \le p < \infty$, by Theorem 2.15, we can approximate $u \in W^{1,p}(V)$ by a sequence of functions $v_l \in C^\infty(V)$, and $C_c^\infty(V) \ni \chi v_l \to \chi \overline{u}$ in $W^{1,p}(V)$. Consequently, the extension $\overline{u} \in W_0^{1,p}(V)$:

$$E : W^{1,p}(U) \to W_0^{1,p}(V) \hookrightarrow W^{1,p}(\mathbb{R}^n), \quad u \mapsto Eu := \overline{u}.$$

(ii) If $p = \infty$, the constant c in (3.1) is actually independent of n.

(iii) If we further assume that ∂U is C^2, then the extension operator $E : u \mapsto \overline{u}$ above is also a bounded linear operator from $W^{2,p}(U)$ to $W^{2,p}(V)$, with

$$\|Eu\|_{W^{2,p}(\mathbb{R}^n)} = \|Eu\|_{W^{2,p}(V)} \le c\|u\|_{W^{2,p}(U)}. \tag{3.4}$$

Theorem 3.2. *Let U be a bounded, open subset of $\mathbb{R}^n$, and let ∂U be Lipschitz. Then $C^{0,1}(\overline{U}) = W^{1,\infty}(U)$.*

Proof. If $u \in C^{0,1}(\overline{U})$, we can apply Step I in the proof of Theorem 2.19 to argue that $u \in W^{1,\infty}(U)$. Conversely, if $u \in W^{1,\infty}(U)$, we can simply apply Step I in the proof of Theorem 2.19 to the extension Eu of u on $\mathbb{R}^n$, which is a convex set. □

3.2 Traces

In this section we discuss the possibility of assigning "boundary values" along aU to a function $W^{1,p}(U)$,

Theorem 3.3 (Trace theorem). *Let $U \subset \mathbb{R}^n$ be a open and bounded set with C^1 boundary. Then there exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U)$ such that*

(i) $Tu = u|_{\partial U}$ if $u \in W^{1,p}(U) \cap C(\overline{U})$, and

(ii) there exists a constant C depending only on U and p such that $\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(U)}$ for all $u \in W^{1,p}(U)$.

*Here T is called the **trace operator**, and Tu is called the **trace** of u on ∂U.*

Proof. $\square$

Theorem 3.4 (Treace-zero functions). *Let $U \subset \mathbb{R}^n$ be a open and bounded set with C^1 boundary, and $u \in W^{1,p}(U)$. Then $u \in W_0^{1,p}(U)$ if and only if $Tu = 0$ on ∂U.*

Proof. $\square$

4 Sobolev Inequalities

4.1 Sub-dimensional Case $p < n$: Gagliardo-Nirenberg-Sobolev Inequality

In this section, we suppose $1 \le p < n$, and we consider the following basic question: Can we estimate the $L^q(\mathbb{R}^n)$-norm of a smooth, compactly supported function in terms of the $L^p(\mathbb{R}^n)$-norm of its derivative. In other words, we are looking for an estimate of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \le c\|Du\|_{L^p(\mathbb{R}^n)}, \quad u \in C_c^\infty(\mathbb{R}^n). \tag{4.1}$$

A scaling argument. We wonder if the estimate (4.1) holds for any $q \in [1, \infty]$. Take $u \in C_c^\infty(\mathbb{R}^n)$ with $u \not\equiv 0$, and define for $\lambda > 0$ the rescaled function $u_\lambda(x) = u(\lambda x)$. Then

$$Du_\lambda = \lambda(Du)_\lambda.$$

We then obtain

$$\|u_\lambda\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u_\lambda|^q \, dx\right)^{1/q} = \left(\lambda^{-n} \int |u|^q \, dx\right)^{1/q} = \lambda^{-n/q} \|u\|_{L^q(\mathbb{R}^n)},$$
$$\|Du_\lambda\|_{L^p(\mathbb{R}^n)} = \left(\sum_{|\alpha|=1} \int_{\mathbb{R}^n} |D^\alpha u|^p\right)^{1/p} = \left(\lambda^{p-n} \sum_{|\alpha|=1} \int_{\mathbb{R}^n} |D^\alpha u|^p\right)^{1/p} = \lambda^{1-n/p} \|Du\|_{L^p(\mathbb{R}^n)}.$$

These norms must scale according to the same exponent, otherwise (4.1) is falsified by letting $\lambda \to 0$ or $\lambda \to \infty$. Hence we have $n/p - n/q = 1$, and $q = \frac{np}{n-p}$.

Definition 4.1 (Sobolev conjugate)**.** If $1 \le p < n$, the Sobolev conjugate of p is

$$p^* = \frac{np}{n-p}.$$

Note that $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$, and $p^* > p$.

We have the following estimate for L^{p^*}-norm f a Sobolev function.

Theorem 4.2 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Assume that $1 \le p < n$. There exists a constant C, depending on p and n only, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in C_c^1(\mathbb{R}^n). \tag{4.2}$$

Proof. Step I: We first prove the case $p = 1$. Since u has compact support, we have

$$u(x) = \int_{-\infty}^{x_i} \partial_{x_i} u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) \, dy_i,$$

We denote by $|Du|_1 = |\partial_{x_1} u| + \cdots + |\partial_{x_n} u|$. Then

$$|u(x)| \le \int_{-\infty}^{x_i} |\partial_{x_i} u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)| \, dy_i \le \int_{-\infty}^{\infty} |Du|_1 \, dx_i.$$

Consequently,

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^n \left(\int_{-\infty}^\infty |Du|_1 \, dx_i\right)^{\frac{1}{n-1}}.$$

We integrate both sides of the last display with respect to the variable x_1. By generalized Hölder's inequality,

$$\begin{aligned}
\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\,dx_1 &\le \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du|_1\,dx_i\right)^{\frac{1}{n-1}} dx_1 \\
&= \left(\int_{-\infty}^{\infty} |Du|_1\,dx_1\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty} |Du|_1\,dx_i\right)^{\frac{1}{n-1}} dx_1 \\
&\le \left(\int_{-\infty}^{\infty} |Du|_1\,dx_1\right)^{\frac{1}{n-1}} \left(\prod_{i=2}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_i\right)^{\frac{1}{n-1}}.
\end{aligned}$$

Again, we integrate both sides with respect to x_2. By generalized Hölder's inequality,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |u(x)|^{\frac{n}{n-1}}\,dx_1\,dx_2 \\
&\le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_2\right)^{\frac{1}{n-1}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |Du|_1\,dx_1\right)^{\frac{1}{n-1}} \left(\prod_{i=3}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_i\right)^{\frac{1}{n-1}} dx_2 \\
&\le \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_2\right)^{\frac{2}{n-1}} \left(\prod_{i=3}^{n} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |Du|_1\,dx_1\,dx_2\,dx_i\right)^{\frac{1}{n-1}}.
\end{aligned}$$

We continue to integrate with respect to $x_3, \cdots, x_n$, and obtain that

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\,dx \le \left(\int_{\mathbb{R}^n} |Du|_1\,dx\right)^{\frac{n}{n-1}}. \tag{4.3}$$

This is indeed the case $p^* = \frac{n}{n-1}$ and $C = 1$ of estimate (4.2).

Step II: Now we consider the case $1 < p < n$. Applying the estimate (4.3) to $v = |u|^\gamma$, where $\gamma > 1$ is to be selected, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^{\frac{\gamma n}{n-1}}\,dx\right)^{\frac{n-1}{n}} &\le \int_{\mathbb{R}^n} \gamma |u|^{\gamma-1} |Du|_1\,dx \\
&\le \gamma \left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |Du|_1^p\,dx\right)^{1/p} \\
&\le \gamma \left(\int_{\mathbb{R}^n} |u|^{\frac{(\gamma-1)p}{p-1}}\,dx\right)^{\frac{p-1}{p}} n^{\frac{p-1}{p}} \|Du\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{4.4}$$

Now we choose $\gamma > 1$ such that $\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1}$. That is, $\gamma = \frac{(n-1)p}{n-p} = \frac{(n-1)p^*}{n}$. Then (4.4) becomes

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\right)^{1/p^*} \le \frac{n^{\frac{p-1}{p}}(n-1)p}{n-p} \|Du\|_{L^p(\mathbb{R}^n)},$$

which completes the proof of (4.2). □

Theorem 4.3 (Estimate for $W^{1,p}$ on $\mathbb{R}^n$, $1 \le p < n$)**.** *Assume that $1 \le p \le n$ and $p \le q \le p^*$, and $u \in W^{1,p}(U)$. Then $u \in L^q(U)$, with the estimate*

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)} \tag{4.5}$$

for some constant C depending only on p, q and n.

Proof. By the Remark under Theorem 2.14, we can find a sequence $u_m \in C_c^\infty(\mathbb{R}^n)$ that converges to u in $W^{1,p}(\mathbb{R}^n)$. According to Theorem 4.2, we have

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq np^* \|Du_m - Du_l\|_{L^p(\mathbb{R}^n)}, \quad \forall l, m \geq 1.$$

Hence (u_m) is a Cauchy sequence in $L^{p^*}(\mathbb{R}^n)$, and $u_m \to \tilde{u}$ for some $\tilde{u} \in L^{p^*}(\mathbb{R}^n)$. Furthermore, $\tilde{u}$ and u are identified, since we can find a subsequence of (u_m) that converges a.e. to $\tilde{u}$ from L^{p^*} convergence, and to u, from L^p convergence. Hence $u \in L^{p^*}(\mathbb{R}^n)$, and

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq np^* \|Du\|_{L^p(\mathbb{R}^n)}.$$

For the estimate (4.5), the case $q = p$ and $q = p^*$ are clear. If $p < q < p^*$, we choose $0 < \theta < 1$ such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$. By Hölder's inequality,

$$\int_{\mathbb{R}^n} |u|^q \, dx = \int_{\mathbb{R}^n} |u|^{\theta q} |u|^{(1-\theta)q} \, dx \leq \left(\int_{\mathbb{R}^n} |u|^p \, dx \right)^{\frac{\theta q}{p}} \left(\int_{\mathbb{R}^n} |u|^{p^*} \, dx \right)^{\frac{(1-\theta)q}{p}}.$$

Therefore

$$\|u\|_{L^q(\mathbb{R}^n)} \leq \|u\|_{L^p(\mathbb{R}^n)}^\theta \|u\|_{L^{p^*}(\mathbb{R}^n)}^{1-\theta} \leq (np^*)^{1-\theta} \|u\|_{L^p(\mathbb{R}^n)}^\theta \|Du\|_{L^p(\mathbb{R}^n)}^{1-\theta}.$$

To derive (4.5), we use Jensen's inequality:

$$\theta \log \frac{a^p}{\theta} + (1-\theta) \log \frac{b^p}{1-\theta} \leq \log(a^p + b^p) \quad \Rightarrow \quad a^\theta b^{1-\theta} \leq \theta^{\frac{\theta}{p}} (1-\theta)^{\frac{1-\theta}{p}} (a^p + b^p)^{1/p}, \qquad \forall a, b > 0.$$

Then we obtain

$$\|u\|_{L^q(\mathbb{R}^n)} \leq (np^*)^{1-\theta} \theta^{\frac{\theta}{p}} (1-\theta)^{\frac{1-\theta}{p}} \left(\|u\|_{L^p(\mathbb{R}^n)}^p + \|Du\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p} =: C \|u\|_{W^{1,p}(\mathbb{R}^n)}.$$

This completes the proof of (4.5). □

Now we give a similar estimate of the $W^{1,p}$-norm of a weakly differentiable function on a Lipschitz domain.

Theorem 4.4 (Estimate for $W^{1,p}$ on Lipschitz domains, $1 \leq p < n$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$ and suppose ∂U is Lipschitz. Assume that $1 \leq p < n$, and $u \in W^{1,p}(U)$. Then $u \in L^{p^*}(U)$, with the estimate*

$$\|u\|_{L^{p^*}(U)} \leq C \|u\|_{W^{1,p}(U)}$$

for some constant C depending only on p, n and U.

Proof. Since ∂U is Lipschitz, by Theorem 3.1, there exists an extension $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\overline{u} = u$ in U, $\overline{u}$ has compact support in $\mathbb{R}^n$, and

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C_1 \|u\|_{W^{1,p}(U)}, \tag{4.6}$$

where C_1 is a constant depending only on p, n and U. Since $\overline{u}$ has compact support, by the Remark under Theorem 2.14, there exists a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that $u_m \to \overline{u}$ in $W^{1,p}(\mathbb{R}^n)$. By Theorem 4.2, $u_m \to \overline{u}$ in $L^{p^*}(\mathbb{R}^n)$ as well, and $\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq np^* \|Du_m\|_{L^p(\mathbb{R}^n)}$. Then we have the limiting bound

$$\|u\|_{L^{p^*}(U)} \leq \underbrace{\|\overline{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq np^* \|D\overline{u}\|_{L^p(\mathbb{R}^n)}}_{m \to \infty} \leq np^* \|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \overset{(4.6)}{\leq} C_1 np^* \|u\|_{W^{1,p}(U)}.$$

The desired result then follows by letting $C = C_1 np^*$. □

Remark. If U is a bounded, open subset of $\mathbb{R}^n$ and ∂U is Lipschitz, we have

$$W^{1,p}(U) \subset L^{p^*}(U) \subset L^q(U), \quad q \in [1, p^*].$$

by Hölder's inequality $\|u\|_{L^q(U)} \le |U|^{\frac{p^*-q}{p^*q}} \|u\|_{L^{p^*}(U)}$, we have

$$\|u\|_{L^q(U)} \le C\|u\|_{W^{1,p}(U)}, \quad q \in [1, p^*],$$

where C is a constant depending only on p, q, n and U.

Theorem 4.5 (Estimate for $W_0^{1,p}$ on bounded domains, $1 \le p < n$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$. Assume that $1 \le p < n$, and $u \in W_0^{1,p}(U)$. Then we have the estimate*

$$\|u\|_{L^q(U)} \le C\|Du\|_{L^p(U)} \tag{4.7}$$

for each $q \in [1, p^*]$, with the constant C depending only on p, q, n and U.

Proof. Since $u \in W_0^{1,p}(U)$, there exists a sequence of functions $u_m \in C_c^\infty(U)$ such that $u_m \to u$ in $W^{1,p}(U)$. We the extend each u_m to $\mathbb{R}^n$ by assigning $u_m = 0$ on $\mathbb{R}^n \backslash U$. By letting $m \to \infty$ in the Gagliardo-Nirenberg-Sobolev inequality for u_m, we obtain

$$\|u\|_{L^{p^*}(U)} \le C\|Du\|_{L^p(U)}.$$

Since U is bounded, we have $|U| < \infty$, and the desired result follows from Hölder's inequality. □

Corollary 4.6 (Classical Poincaré's inequality)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$, and $1 \le p \le \infty$. For any $u \in W_0^{1,p}(U)$, we have the estimate*

$$\|u\|_{L^p(U)} \le C\|Du\|_{L^p(U)}, \tag{4.8}$$

where the constant C depending only on p, n and U.

Proof. For $1 \le p < n$, the estimate (4.8) is a special case of (4.7), since $p < p^*$. For $n \le p < \infty$, we choose $1 \le q < n$ such that $q < n \le p < q^* := \frac{nq}{n-q}$. Since $W_0^{1,p}(U) \subset W^{1,q}(U)$, by (4.7), we have

$$\|u\|_{L^p(U)} \le C\|Du\|_{L^q(U)} \le |U|^{\frac{pq}{p-q}} C\|Du\|_{L^p(U)}.$$

Finally, for $p = \infty$, we take a sequence $u_m \in C_c^\infty(U)$ that converges to u in $W^{1,\infty}(U)$. Using the fundamental theorem of calculus, we have

$$\begin{aligned} |u_m(x_1, \cdots, x_n)| &= \left| \int_{-\infty}^{x_i} \partial_{x_i} u_m(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)\, dy_i \right| \\ &\le \int_{-\infty}^{\infty} \|Du_m\|_{L^\infty(U)}\, dx_i \le \operatorname{diam}(U)\, \|Du_m\|_{L^\infty(U)} \end{aligned}$$

By taking the supremum of the left hand side and letting $m \to \infty$ in the last display, we can obtain that $\|u\|_{L^\infty(U)} \le \operatorname{diam}(U)\, \|Du\|_{L^\infty(U)}$. This complete the proof. □

The borderline case: $p = n$**.** Owing to Theorem 4.5 and the fact that $p^* = \frac{np}{n-p} \to \infty$ as $p \nearrow n$, we might expect $u \in L^\infty(U)$, provided $u \in W^{1,n}(U)$. This is however false if $n > 1$.

As a counterexample, let $U = B^0(0,1)$ be the unit open ball in $\mathbb{R}^n$, where $n > 1$. Then the function $u(x) = \log\log\big(1 + \frac{1}{|x|}\big)$ belongs to $W^{1,n}(U)$, but not to $L^\infty(U)$.

4.2 Super-dimensional Case $p > n$: Morrey's Inequality

In this section, we assume that $n < p \le \infty$. We show that u has a Hölder continuous representative, provided that $u \in W^{1,p}(U)$.

Theorem 4.7 (Morrey's inequality)**.** *Assume that $n < p \le \infty$. There exists a constant C, depending on p and n only, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall u \in C^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \tag{4.9}$$

where $\gamma = 1 - \frac{n}{p}$.

Proof. Step I: We claim that there exists a constant C_1, depending only on n, such that

$$\frac{1}{\mathcal{L}^n(B(x,r))}\int_{B(x,r)} |u(y) - u(x)|\, dy \le C_1 \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy, \tag{4.10}$$

for each ball $B(x,r)$, where $\mathcal{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$. To this end, take any $|w| = 1$. If $0 < s < r$,

$$|u(x+sw) - u(x)| = \left|\int_0^s \frac{d}{dt}u(x+tw)\, dt\right| = \left|\int_0^s Du(x+tw)\cdot w\, dt\right| \le \int_0^s |Du(x+tw)|\, dt.$$

Integrate with respect to w on $\partial B(0,1)$:

$$\begin{aligned}
\int_{\partial B(0,1)} |u(x+sw) - u(x)|\, dS(w) &\le \int_0^s \int_{\partial B(0,1)} |Du(x+tw)|\, dS(w)\, dt \\
&\overset{y=x+tw}{=} \int_0^s \int_{\partial B(x,t)} \frac{|Du(y)|}{t^{n-1}}\, dS(y)\, dt \\
&\overset{t=|x-y|}{=} \int_{B(x,s)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy = \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy.
\end{aligned}$$

By changing the variable $z = x + sw$ in the left hand side of the last display, we have

$$\int_{\partial B(x,s)} |u(z) - u(x)| dS(z) \le s^{n-1}\int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy.$$

Next integrate with respect to s from 0 to r:

$$\int_{B(x,r)} |u(y) - u(x)|\, dy \le \frac{r^n}{n}\int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy.$$

This completes the proof of (4.10).

Step II: Fix any $x \in \mathbb{R}^n$. By (4.10) and Hölder's inequality,

$$\begin{aligned}
|u(x)| &\le \frac{1}{\mathcal{L}^n(B(x,1))}\left(\int_{B(x,1)} |u(x) - u(y)|\, dy + \int_{B(x,1)} |u(y)|\, dy\right) \\
&\le C_1 \int_{B(x,1)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy + \mathcal{L}^n(B(x,1))^{-1/p}\|u\|_{L^p(B(x,1))} \\
&\le C_1 \left(\int_{\mathbb{R}^n} |Du|^p\, dy\right)^{1/p} \left(\int_{B(x,1)} |y-x|^{-\frac{(n-1)p}{p-1}}\, dy\right)^{\frac{p-1}{p}} + \mathcal{L}^n(B(x,1))^{-1/p}\|u\|_{L^p(\mathbb{R}^n)} \\
&\le C\|u\|_{W^{1,p}(\mathbb{R}^n)},
\end{aligned}$$

where $C = C(n,p)$ is a constant. The last estimate holds since $p > n$ implies $(n-1)\frac{p}{p-1} < n$, and

$$\int_{B(x,1)} |y-x|^{-\frac{(n-1)p}{p-1}}\, dy < \infty.$$

Step III: Choose any two points $x, y \in \mathbb{R}^n$, and write $r := |x-y|$. Let $W = B(x,r) \cap B(y,r)$. Then

$$|u(y) - u(x)| \le \frac{1}{\mathcal{L}^n(W)} \left(\int_W |u(x) - u(z)|\, dz + \int_W |u(y) - u(z)|\, dz \right).$$

By estimate (4.10), we have

$$\begin{aligned}
\frac{1}{\mathcal{L}^n(W)} \int_W |u(x) - u(z)|\, dz &\le \frac{\mathcal{L}^n(B(x,r))}{\mathcal{L}^n(W)} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} |u(x) - u(z)|\, dz \\
&\le \frac{C_1 \mathcal{L}^n(B(x,r))}{\mathcal{L}^n(W)} \int_{B(x,r)} \frac{|Du(z)|}{|z-x|^{n-1}}\, dz \\
&\le \frac{C_1 \mathcal{L}^n(B(x,r))}{\mathcal{L}^n(W)} \left(\int_{B(x,r)} |Du|^p\, dz \right)^{1/p} \left(\int_{B(x,r)} \frac{dz}{|z-x|^{\frac{(n-1)p}{p-1}}} \right)^{\frac{p-1}{p}} \\
&\le C_2 \left(r^{n - \frac{(n-1)p}{p-1}} \right)^{\frac{p-1}{p}} \|Du\|_{L^p(B(x,r))} \le C_2 r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where C_2 is a constant depending on n and p only. Similarly, we have

$$\frac{1}{\mathcal{L}^n(W)} \int_W |u(x) - u(z)|\, dz \le C_2 r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}.$$

Consequently,

$$[u]_{C^{0,1-\frac{n}{p}}(\mathbb{R}^n)} = \sup_{x \ne y} \frac{|u(y) - u(x)|}{|y-x|^{1-\frac{n}{p}}} \le C \|Du\|_{L^p(\mathbb{R}^n)}.$$

This inequality together with (4.2) completes the proof of (4.9). $\square$

Remark. We provide a slight variant of the estimate of $|u(x) - u(y)|$, where $|x-y| \le r$. Since both $B(x,r)$ and $B(y,r)$ are include in the ball $B(x,2r)$, we have

$$|u(y) - u(x)| \le C r^{1-\frac{n}{p}} \|Du\|_{L^p(B(x,2r))}$$

for all $u \in C^1(B(x,2r))$, $y \in B(x,r)$ and $n < p < \infty$.

Theorem 4.8 (Estimate for $W^{1,p}$ on Lipschitz domains, $n < p \le \infty$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$, and suppose that ∂U is Lipschitz. Assume $n < p \le \infty$ and $u \in W^{1,p}(U)$. Then u has a representative $u^* \in C^{0,\gamma}(\overline{U})$ for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \le C \|u\|_{W^{1,p}(U)}, \tag{4.11}$$

where the constant C depends on p, n and U only.

Proof. The case $p = \infty$ can be easily adapted from Theorem 3.2. Hence we assume that $n < p < \infty$.

Since ∂U is Lipschitz, by Theorem 3.1, there exists an extension $\overline{u} \in W^{1,p}(\mathbb{R}^n)$ such that $\overline{u} = u$ a.e. in U, $\overline{u}$ has compact support in $\mathbb{R}^n$, and

$$\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C_1 \|u\|_{W^{1,p}(U)}, \tag{4.12}$$

where C_1 is a constant depending only on p, n and U. According to the Remark under Theorem 2.14, we can find a sequence of functions $u_m \in C_c^\infty(\mathbb{R}^n)$ converging to $\overline{u}$ in $W^{1,p}(\mathbb{R}^n)$. By Theorem 4.7, (u_m) is also a Cauchy sequence in $C^{1-\frac{n}{p}}(\mathbb{R}^n)$, which converges to some $u^* \in C^{1-\frac{n}{p}}(\mathbb{R}^n)$. Clearly, $u^* = u$ a.e. on U. Furthermore, letting $m \to \infty$ in Morrey's inequality for u_m yields $\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|\overline{u}\|_{W^{1,p}(\mathbb{R}^n)}$. Combining this with estimate (4.12) concludes the proof. □

Remark. The preceding proof remains valid if we replace U by $\mathbb{R}^n$ and omit the extension step. We therefore restate our conclusion as follows: Assume $n < p \leq \infty$ and $u \in W^{1,p}(\mathbb{R}^n)$. Then u has a representative $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ for $\gamma = 1 - \frac{n}{p}$, with the estimate

$$\|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

where the constant C depends on p and n only.

Now we use the tool of Morrey's inequality to investigate more closely the connections between weak partial derivatives and partial derivatives.

Theorem 4.9 (Super-dimensional differentiability almost everywhere)**.** *Assume that $u \in W^{1,p}_{\text{loc}}(U)$ for some $n < p \leq \infty$. Then u is differentiable a.e. in U, and its gradient equals its weak gradient a.e..*

Proof. We first assume that $n < p < \infty$. We identify u to its continuous version by applying Morrey's inequality on a countable set of balls covering U. For a.e. $x \in U$, by Lebesgue's differentiation theorem,

$$\frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} |Du(x) - Du(z)|^p \, dz \to 0 \quad as \quad r \to 0.$$

We then fix such a point x, and set $v(y) := u(y) - u(x) - Du(x) \cdot (y - x)$. Since the differentiation is a local problem, we choose $B(x, \delta) \subset U$. Then $v \in W^{1,p}(B(x,\delta))$.

By Proposition 1.8 and Theorem 2.14, the mollifications $v^\epsilon \in C^\infty(U)$ converges to v uniformly on $B(x,\delta)$ and in $W^{1,p}(B(x,\delta))$ as $\epsilon \to 0$. According to the remark under Theorem 4.7 and by approximation $\epsilon \to 0$, for each $y \in U$ with $r := |x - y| < \delta/2$, we have Morrey's estimate

$$|v(y) - v(x)| \leq Cr^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |Dv(z)|^p \, dz \right)^{1/p}.$$

Consequently,

$$\begin{aligned} |u(y) - u(x) - Du(x) \cdot (y - x)| &\leq Cr^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |Du(x) - Du(z)|^p \, dz \right)^{1/p} \\ &\leq C' r \left(\frac{1}{\mathcal{L}^n(B(x,2r))} \int_{B(x,2r)} |Du(x) - Du(z)|^p \, dz \right)^{1/p} = o(r) = o(|x - y|). \end{aligned}$$

Hence u is differentiable at x, and its gradient coincides its weak gradient at x. Finally, for the case $p = \infty$, just note that $W^{1,\infty}_{\text{loc}}(U) \subset W^{1,p}_{\text{loc}}(U)$ for all $1 \leq p < \infty$. □

The following theorem is a direct consequence of Theorem 4.9.

Theorem 4.10 (Rademacher's theorem)**.** *Let u be locally Lipschitz continuous in U. Then u is differentiable almost everywhere in U.*

4.3 General Sobolev Inequalities

4.3.1 Sub-dimensional Case: $kp < n$

Theorem 4.11 (General Sobolev inequality, $kp < n$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$, with a Lipschitz boundary. Assume $u \in W^{k,p}(U)$, and $kp < n$. Then $u \in L^q(U)$, where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}, \quad q = \frac{np}{n-kp}.$$

Furthermore, we have the estimate

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)},$$

where C is a constant depending only on k, p, n and U.

Proof. Step I: For every multi-index $|\alpha| \leq k-1$, we have $D^\alpha u \in W^{1,p}(U)$. By Gagliardo-Nirenberg-Sobolev inequality [Theorem 4.4], there exists a constant $C = C(n,p,U) > 0$ depending only on n, p and U, such that

$$\|D^\alpha u\|_{L^{p^*}(U)} \leq C\|D^\alpha u\|_{W^{1,p}(U)} \leq C\|u\|_{W^{k,p}(U)}.$$

Hence $u \in W^{k-1,p^*}(U)$, where $p < p^* = \frac{np}{n-p} < n$. If $k = 2$, we are done by applying Gagliardo-Nirenberg-Sobolev inequality once again, where $q = p^{**} = \frac{np^*}{n-p^*} = \frac{np}{n-2p}$:

$$\|u\|_{L^{p^{**}}(U)} \leq C(n,p^*,U)\|u\|_{W^{1,p^*}(U)} \leq C(n,p^*,U)(1+n)C(n,p,U)\|u\|_{W^{2,p}(U)}.$$

Step II: We denote $p_2 = p^{**}$, $p_3 = p^{***}$, and so on. If $k \geq 3$, we can prove by induction such that

$$\begin{aligned}
&\|D^\alpha u\|_{L^{p^{**}}(U)} \leq C_2\|D^\alpha u\|_{W^{1,p^*}(U)} \leq C_2\|u\|_{W^{k-1,p^*}(U)}, \quad \forall|\alpha| \leq k-2, \quad \textit{and} \ \ u \in W^{k-2,p^{**}}(U);\\
&\|D^\alpha u\|_{L^{p^{***}}(U)} \leq C_3\|D^\alpha u\|_{W^{1,p^{**}}(U)} \leq C_3\|u\|_{W^{k-2,p^{**}}(U)}, \quad \forall|\alpha| \leq k-3, \quad \textit{and} \ \ u \in W^{k-3,p^{***}}(U);\\
&\cdots;\\
&\|D^\alpha u\|_{L^{p_{k-1}}(U)} \leq C_{k-1}\|D^\alpha u\|_{W^{1,p_{k-2}}(U)} \leq C_{k-1}\|u\|_{W^{2,p_{k-2}}(U)}, \quad \forall|\alpha| \leq 1, \quad \textit{and}\ u \in W^{1,p_{k-1}}(U).
\end{aligned}$$

Hence $u \in W^{1,p_{k-1}}(U)$. Since $p < p_{k-1} < n$, again by Gagliardo-Nirenberg-Sobolev inequality, we have

$$\begin{aligned}
\|u\|_{L^{p_k}(U)} \leq C_k\|u\|_{W^{1,p_{k-1}}(U)} &\leq (1+n)C_kC_{k-1}\|u\|_{W^{2,p_{k-2}}(U)}\\
&\leq (1+n)\left(1+n+n^2\right)C_kC_{k-1}C_{k-2}\|u\|_{W^{3,p_{k-3}}(U)} \leq \cdots\\
&\leq (1+n)\left(1+n+n^2\right)\cdots\left(1+n+n^2+\cdots+n^{k-1}\right)C_kC_{k-1}\cdots C_1\|u\|_{W^{k,p}(U)}.
\end{aligned}$$

where $C_1, \cdots, C_k$ are constants depending only on k, n, p and U. This completes the proof. $\square$

Remark. In fact, we have the inclusions

$$W^{k,p}(U) \subset W^{k-1,p^*}(U) \subset W^{k-2,p^{**}}(U) \subset \cdots \subset W^{k-l,q}(U),$$

where $l \in \{0, 1, \cdots, k\}$ and $\frac{1}{q} = \frac{1}{p} - \frac{l}{n}$. Moreover, there exists a constant C depending only on n, p, q, l and U such that

$$\|u\|_{W^{k-l,q}(U)} \leq C\|u\|_{W^{k,p}(U)}, \quad \forall u \in W^{k,p}(U).$$

This means that $W^{k,p}(U) \hookrightarrow W^{k-l,q}(U)$ is a continuous embedding, where $q = \frac{np}{n-lp} > p$.

4.3.2 Super-dimensional Case: $kp > n$

Theorem 4.12 (General Sobolev inequality, $kp > n$)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$, with a Lipschitz boundary. Assume $u \in W^{k,p}(U)$, and $kp > n$. Then u has a representative $u^* \in C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(\overline{U})$, where*

$$\gamma = \begin{cases} 1 + \left\lfloor \frac{n}{p} \right\rfloor - \frac{n}{p}, & \frac{n}{p} \notin \mathbb{N}, \\ \text{any } \mu \in (0,1), & \frac{n}{p} \in \mathbb{N}. \end{cases}$$

Furthermore, we have the estimate

$$\|u^*\|_{C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}(U)} \le C\|u\|_{W^{k,p}(U)},$$

where C is a constant depending only on k, p, n, γ and U.

Proof. Case I: $n/p \notin \mathbb{N}$. The key idea is to apply general Sobolev inequality [Theorem 4.11] to the largest sub-dimensional case $lp < n$. Given $lp < n$, we have $u \in W^{k-l,r}(U)$, where $\frac{1}{r} = \frac{1}{p} - \frac{l}{n}$. Choose $l \in \mathbb{N}$ such that $l < \frac{n}{p} < l + 1$, that is, $l = \lfloor n/p \rfloor$. Then $r = \frac{np}{n-pl} > n$ is super-dimensional, $k - l \ge 1$, and $D^\alpha u \in W^{1,r}(U)$ admits a representative $(D^\alpha u)^* \in C^{0,\gamma}(\overline{U})$ by Morrey's inequality for each $|\alpha| \le k - l - 1$, where $\gamma = 1 - n/r = 1 + \lfloor n/p \rfloor - n/p$. Furthermore, we have the estimate

$$\|D^\alpha u\|_{C^{0,\gamma}(\overline{U})} \le C\|D^\alpha u\|_{W^{1,r}(U)} \le C\|u\|_{W^{k-l,r}(U)},$$

where the constant C only depends on n, p and U. Consequently, $u^* \in C^{k-\lceil \frac{n}{p} \rceil,\gamma}(\overline{U})$, and

$$\|u\|_{C^{k-l-1,\gamma}(\overline{U})} = \sum_{|\alpha| \le k-l-1} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha|=k-l-1} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} \le C'\|u\|_{W^{k-l,r}(U)},$$

where the constant C' only depends on n, p, k and U.

Case II: $n/p \in \mathbb{N}$. To apply general Sobolev inequality [Theorem 4.11] to the sub-dimensional case, we choose $l = \frac{n}{p} - 1 \in \{0, 1, \cdots, k-2\}$. Then $u \in W^{k-l,q}(U)$ for $q = \frac{np}{n-lp} = n$. By Gagliardo-Nirenberg-Sobolev inequality, for all $r \in (n, \infty)$, we have

$$\|D^\alpha u\|_{L^r(U)} \le C\|D^\alpha u\|_{W^{1,\frac{nr}{n+r}}(U)}, \quad \forall |\alpha| \le k - l - 1 = k - \frac{n}{p},$$

where C is a constant depending only on n, r and U, and $D^\alpha u \in L^r(U)$. By Morrey's inequality, we have $D^\alpha u \in C^{0,1-\frac{n}{r}}(\overline{U})$ for all $|\alpha| \le k - \frac{n}{p} - 1$ and all $r \in (n, \infty)$. Consequently, $u \in C^{k-\frac{n}{p}-1,\gamma}(\overline{U})$ for all $0 < \gamma < 1$, and we have the estimate

$$\|u\|_{C^{k-\frac{n}{p}-1,\gamma}(\overline{U})} \le C'\|u\|_{W^{k-l,n}(U)} \le C''\|u\|_{W^{k,p}(U)},$$

where C' is a constant depending only on k, n, p, γ and U. $\square$

Remark. For the case $p = \infty$, we have the limit conclusion $W^{1,\infty}(U) = C^{0,1}(\overline{U})$ Theorem 3.2 for $k = 1$.

4.3.3 The Borderline Case: $kp = n$

Lemma 4.13. *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Let*

$$\begin{cases} p = \infty, & n = 1, \\ 1 \leq p < \infty, & n \geq 2. \end{cases}$$

Then $W^{1,n}(U) \subset L^p(U)$, and there exists a constant C, depending on n, p and U only, such that

$$\|u\|_{L^p(U)} \leq C\|u\|_{W^{1,n}(U)}, \quad \forall u \in W^{1,n}(U).$$

Proof. Case I: $n = 1$. If $v \in C_c^\infty(\mathbb{R})$, we have

$$|v(x)| \leq \int_{-\infty}^{\infty} |Du(y)| dy.$$

Hence $\|v\|_{L^\infty(\mathbb{R})} \leq \|Dv\|_{L^1(\mathbb{R})} \leq \|v\|_{W^{1,1}(\mathbb{R})}$. Then for each $u \in W^{1,1}(U)$, extend u to $\overline{u} \in W^{1,1}(\mathbb{R})$ with

$$\|\overline{u}\|_{W^{1,1}(\mathbb{R})} \leq c\|u\|_{W^{1,1}(U)},$$

where c is a constant depending on U only. By approximation $\overline{u}$ with $C_c^\infty(\mathbb{R})$, we have

$$\|u\|_{L^\infty(U)} \leq \|\overline{u}\|_{L^\infty(\mathbb{R})} \leq \|\overline{u}\|_{W^{1,1}(\mathbb{R})} \leq c\|u\|_{W^{1,1}(U)}.$$

Case II: $n \geq 2$. Take $n \leq q < \infty$, and set $\frac{1}{s} = \frac{1}{n} + \frac{1}{q}$. Then $1 \leq s < n$, and $q = \frac{ns}{n-s}$. Since U is bounded, by Hölder's inequality, we have

$$\|u\|_{W^{1,s}(U)} \leq (1+n)^{\frac{1}{n}-\frac{1}{s}} |U|^{\frac{n-s}{ns}} \|u\|_{W^{1,n}(U)}.$$

Since $q = s^* = \frac{ns}{n-s}$, by Theorem 4.4, we can find a constant $C(n, q, U)$ such that

$$\|u\|_{L^q(U)} \leq C(n, q, U)\|u\|_{W^{1,s}(U)} \leq C'(n, q, U)\|u\|_{W^{1,n}(U)}.$$

Since $|U| < \infty$, we have

$$\|u\|_{L^p(U)} \leq C''(n, q, U)\|u\|_{W^{1,n}(U)}$$

for all $1 \leq q \leq p$. Since q can be chosen arbitrarily large, the result follows. $\square$

Remark. The conclusion still holds if $n = 1$ and we replace U by $\mathbb{R}$, where constant C is 1.

Theorem 4.14. *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Assume $u \in W^{k,p}(U)$, and $kp = n$. Then $u \in L^q(U)$ for all $1 \leq q < \infty$, and we have the estimate*

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{k,p}(U)},$$

where C is a constant depending only on k, p, q, n and U.

Proof. Similar to our proof of Theorem 4.12, we have the inclusions

$$W^{k,p}(U) \subset W^{k-1,p^*}(U) \subset W^{k-2,p^{**}}(U) \subset \cdots \subset W^{1,n}(U).$$

The last inclusion holds since $\frac{1}{n} = \frac{1}{p} - \frac{k-1}{n}$. The result then immediately follows from Lemma 4.13. $\square$

4.4 Compact Embeddings: Rellich-Kondrachov Compactness Theorem

The Gagliardo-Nirenberg-Sobolev inequality shows that $W^{1,p}(U)$ is continuously embedded into $L^{p^*}(U)$ in the sub-dimensional case $1 \leq p < n$. Next, we are going to demonstrate that $W^{1,p}(U)$ is in fact compactly embedded into the space $L^q(U)$ when $1 \leq q < p^*$.

Definition 4.15 (Compact Embedding)**.** Let X and Y be Banach spaces, and $X \subset Y$. We say X is *compactly embedded* in Y, written $X \Subset Y$, if the identity operator

$$\mathrm{Id} : X \to Y, \quad x \mapsto x$$

is continuous and compact, i.e.

(i) there exist some constant c such that $\|x\|_Y \leq c\|x\|_X$ for all $x \in X$, and

(ii) each bounded subset of X is precompact in Y.

Remark. Since compactness coincides sequential compactness in metrizable spaces, (ii) equals that *every bounded sequence of points of X has a subsequence converging in Y.*

Theorem 4.16 (Rellich-Kondrachov Compactness Theorem)**.** *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Assume $1 \leq p < n$. Then*

$$W^{1,p}(U) \Subset L^q(U)$$

for all $1 \leq q < p^*$.

Proof. Step I: Assume that $1 \leq q < p^*$. Using Gagliardo-Nirenberg-Sobolev inequality [Theorem 4.4], we obtain the continuous embedding $W^{1,p}(U) \hookrightarrow L^q(U)$, with

$$\|u\|_{L^q(U)} \leq C\|u\|_{W^{1,p}(U)}$$

for all $u \in W^{1,p}(U)$, where the constant C depending only on n, p, q and U. Then it remains to show that any bounded sequence (u_m) in $W^{1,p}(U)$ has a subsequence (u_{m_l}) converging in $L^q(U)$.

Step II: By extension theorem [3.1], we may assume that every u_m is in $W^{1,p}(\mathbb{R}^n)$ and supported on a precompact set $V \Subset U$, and $\sup_{m\in\mathbb{N}} \|u_m\|_{W^{1,p}(\mathbb{R}^n)} < \infty$.

Then we study the mollifiers $u_m^\epsilon = \eta_\epsilon * u_m$, and we may assume that the support of u_m^ϵ is in V for all $m \in \mathbb{N}$. We first prove that

$$\lim_{\epsilon\to 0} \sup_{m\in\mathbb{N}} \|u_m^\epsilon - u_m\|_{L^q(V)} = 0. \tag{4.13}$$

If u_m is smooth, we have

$$\begin{aligned} u_m^\epsilon(x) - u_m(x) &= \frac{1}{\epsilon^n} \int_{B(x,\epsilon)} \eta\left(\frac{x-z}{\epsilon}\right) (u_m(z) - u_m(x))\, dz \\ &= \int_{B(0,1)} \eta(y) \left(u_m(x-\epsilon y) - u_m(x)\right)\, dy \\ &= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt} \left(u_m(x-\epsilon t y)\right) dt\, dy \\ &= -\epsilon \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\epsilon t y) \cdot y\, dt\, dy. \end{aligned}$$

Consequently,

$$\begin{aligned}\|u_m^\epsilon - u_m\|_{L^1(V)} &= \int_V |u_m^\epsilon(x) - u_m(x)|\,dx \\ &\le \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x-\epsilon ty)|\,dx\,dt\,dy \\ &\le \epsilon \int_V |Du_m(z)|\,dz = \epsilon\|Du_m\|_{L^1(V)}.\end{aligned}$$

By approximation, this estimate also holds for $u_m \in W^{1,p}(U)$. Since V is bounded, we have

$$\|u_m^\epsilon - u_m\|_{L^1(V)} \le \epsilon\|Du_m\|_{L^1(V)} \le \epsilon C\|Du_m\|_{L^p(V)}$$

Note that u_m is bounded in $W^{1,p}(\mathbb{R}^n)$. Then the estimate (4.13) holds when $q=1$. If $1<q<p^*$, let $0<\theta<1$ be such that

$$\frac{\theta}{1} + \frac{1-\theta}{p^*} = \frac{1}{q}.$$

Akin to the interpolation statement employed in the proof of Theorem 4.3, we have

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \le \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta}.$$

While the first term converges to 0, the estimate (4.13) follows from the boundedness of the second term, by Gagliardo-Nirenberg-Sobolev inequality.

Step III: Fix any $\epsilon>0$. We verify that $(u_m^\epsilon)_{m=1}^\infty$ satisfies Arzelà-Ascoli criterion: We claim that the sequence $(u_m^\epsilon)_{m=1}^\infty$ is uniformly bounded and uniformly equicontinuous, i.e.

(i) $\sup_{m\in\mathbb{N}} \|u_m^\epsilon\|_\infty < \infty$, and

(ii) for all $\eta>0$, there exists $\delta>0$ such that for all $m\in\mathbb{N}$ and all $|x-y|<\delta$, $|u_m^\epsilon(x)-u_m^\epsilon(y)|<\eta$.

To prove the first assertion, note that

$$\begin{aligned}|u_m^\epsilon(x)| &\le \int_{B(x,\epsilon)} \eta_\epsilon(x-y)|u_m(y)|\,dy \le \|\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \\ &\le \frac{1}{\epsilon^n}\|u_m\|_{L^1(V)} \le \frac{|V|^{1/p}}{\epsilon^n}\|u_m\|_{L^p(V)}.\end{aligned}$$

Since $(u_m)_{m=1}^\infty$ is bounded in $W^{1,p}(U)$, the first assertion holds. For the second assertion,

$$\begin{aligned}|Du_m^\epsilon(x)| &\le \int_{B(x,\epsilon)} |D\eta_\epsilon(x-y)|\,|u_m(y)|\,dy \\ &\le \|D\eta_\epsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \le \frac{|V|^{1/p}}{\epsilon^{1+n}}\|Du_m\|_{L^p(V)}.\end{aligned}$$

Consequently, we have $\sup_{m\in\mathbb{N}}\|Du_m^\epsilon\|_{L^\infty(V)} < \frac{C}{\epsilon^{1+n}}$ for some constant C depending only on n,p and V, and the second assertion holds. By Arzelà-Ascoli theorem, the sequence $(u_m^\epsilon)_{m=1}^\infty$ has a subsequence $(u_m^j)_{j=1}^\infty$ that converges uniformly on V, and

$$\limsup_{j,k\to\infty} \left\|u_{m_j}^\epsilon - u_{m_k}^\epsilon\right\|_{L^q(V)} = 0. \tag{4.14}$$

Step IV: Fix any $\delta>0$. By estimate (4.13), we choose $\epsilon>0$ to so small that

$$\sup_{m\in\mathbb{N}} \|u_m^\epsilon - u_m\|_{L^q(V)} < \frac{\delta}{2}.$$

Combining this bound with (4.14), we obtain

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \le \limsup_{j,k\to\infty} \left(\|u_{m_j} - u^{\epsilon}_{m_j}\|_{L^q(V)} + \|u^{\epsilon}_{m_j} - u^{\epsilon}_{m_k}\|_{L^q(V)} + \|u^{\epsilon}_{m_k} - u_{m_k}\|_{L^q(V)} \right) < \delta,$$

where $(m_j)_{j=1}^{\infty}$ is the subsequence chosen in Step III, which depends on ϵ. Next, we employ our conclusion on $\delta = 1, \frac{1}{2}, \frac{1}{3}, \cdots$ and use Cantor's standard diagonal statement to extract a subsequence $(m_l)_{l=1}^{\infty}$ satisfying

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

By completeness of the space $L^q(V)$, the result follows. $\square$

For $n < p \le \infty$, we have a similar conclusion following from Morrey's inequality and Arzelà-Ascoli theorem.

Theorem 4.17. *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Assume $n < p \le \infty$. Then*

$$W^{1,p}(U) \Subset L^q(U)$$

for all $1 \le q \le \infty$.

Proof. By Arzelà-Ascoli theorem, we know that $C^{0,\gamma}(\overline{U}) \Subset C(\overline{U})$ for all $0 < \gamma \le 1$. Let $(u_m)_{m=1}^{\infty}$ be a bounded sequence in $W^{1,p}(U)$. By Morrey's inequality, (u_m), identified to its Hölder continuous version, is also bounded in $C^{0,1-\frac{n}{p}}(\overline{U})$. Hence there is a subsequence $(u_{m_k})_{k=1}^{\infty}$ that converges uniformly on U. Since U is bounded, $(u_{m_k})_{k=1}^{\infty}$ converges in $L^q(U)$ for all $1 \le q \le \infty$, and the result follows. $\square$

For the borderline case $p = n$, we have the following limiting conclusion.

Theorem 4.18. *Let U be a bounded, open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Then*

$$W^{1,n}(U) \Subset L^q(U)$$

for all $1 \le q < \infty$.

Proof. According to Lemma 4.13, the embedding $W^{1,n}(U) \hookrightarrow L^q(U)$ is continuous for all $1 \le q < \infty$. Now take any bounded sequence $(u_m)_{m=1}^{\infty}$ in $W^{1,n}(U)$. Then for every $1 \le p < n$, since U is bounded, $(u_m)_{m=1}^{\infty}$ is also bounded in $W^{1,p}(U)$. By Rellich-Kondrachov compactness theorem, for any $1 \le q < p^*$, there exists a subsequence $(u_{m_k})_{k=1}^{\infty}$ that converges in $L^q(U)$. Since $p^* = \frac{np}{n-p} \to \infty$ as $p \to n$, the result follows. $\square$

Remark. Summarizing Theorems 4.16, 4.17 and 4.18, we have

$$W^{1,p}(U) \Subset L^p(U)$$

for all $1 \le p \le \infty$. Moreover, we have

$$W^{1,p}_0(U) \Subset L^p(U)$$

for all $1 \le p \le \infty$, even if ∂U is not Lipschitz.

4.5 Poincaré's Inequality

Notation. Given a bounded set $U \subset \mathbb{R}^n$ and a function $u \in L^1(U)$, define the *mean value of u in U* as

$$(u)_U = \frac{1}{|U|} \int_U u(x)\,dx.$$

Similarly, define the *mean value of* $u \in L^1(B(x,r))$ over the ball $B(x,r)$ as

$$(u)_{x,r} = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\,dy.$$

Theorem 4.19 (Poincaré's inequality)**.** *Let U be a bounded, open and connected subset of $\mathbb{R}^n$, with a Lipschitz boundary. Assume $1 \le p \le \infty$. Then there exists a constant C, depending only on n, p and U, such that*

$$\|u - (u)_U\|_{L^p(U)} \le C\|Du\|_{L^p(U)}$$

for each $u \in W^{1,p}(U)$.

Proof. Argue by contradiction. Were the estimate false, there would exist for each $m \in \mathbb{N}$ a Sobolev function $u_m \in W^{1,p}(U)$ satisfying

$$\|u_m - (u_m)_U\|_{L^p(U)} > m\|Du_m\|_{L^p(U)}.$$

We then renormalize by defining

$$v_m = \frac{u_m - (u_m)_U}{\|u_m - (u_m)_U\|_{L^p(U)}}, \quad m = 1, 2, \cdots.$$

Thus $(v_m)_U = 0$, $\|v_m\|_{L^p(U)} = 1$, and $\|Dv_m\|_{L^p(U)} \le \frac{1}{m}$. In particular, the sequence $(v_m)_{m=1}^\infty$ is bounded in $W^{1,p}(U)$. By Rellich-Kondrachov compactness theorem, there is a subsequence $(v_{m_k})_{k=1}^\infty$ that converges in $L^p(U)$, with the limit written by $v \in L^p(U)$. Clearly, we have $(v)_U = 0$, and $\|v\|_{L^p(U)} = 1$. On the other hand, for each $\phi \in C_c^\infty(U)$, one have

$$\int_U v\partial_{x_i}\phi\,dx = \lim_{k\to\infty} \int_U v_{m_k}\partial_{x_i}\phi\,dx = \lim_{k\to\infty} \int_U (D_{x_i}v_{m_k})\phi\,dx = 0, \quad i = 1, \cdots, n.$$

Therefore, $v \in W^{1,p}(U)$, and $Dv = 0$ a.e. on U.

Now we prove that v is constant a.e. on U. Given $\epsilon > 0$, we take the mollification $v^\epsilon = \eta_\epsilon * v$. Clearly, $D_{x_i}v^\epsilon = \eta_\epsilon * D_{x_i}v = 0$ on U^ϵ for all $i = 1, \cdots, n$. Consequently, v^ϵ remains constant on each connected component of U^ϵ. Next, given any $x, y \in U$, since U is connected, we can connect them with a polygonal path $\Gamma \subset U$. Let $\delta = \inf_{z\in\Gamma} d(z, \partial U)$, and take $\epsilon < \delta/2$. Then $\Gamma \subset U^\epsilon$, and x, y lies in the same component of U^ϵ. Hence $v^\epsilon(x) = v^\epsilon(y)$ for all $\epsilon < \delta/2$. By Proposition 1.8, since $v^\epsilon \to v$ a.e. on U, we obtain that v is constant a.e. on U. Finally, since $(v)_U = 0$, we have $v \equiv 0$. However, this implies $\|v\|_{L^p(U)} = 0$, a contradiction! □

We immediately obtain the following result.

Theorem 4.20 (Poincaré's inequality for a ball)**.** *Assume $1 \le p \le \infty$. Then there exists a constant C, depending only on n and p, such that*

$$\|u - (u)_{x,r}\|_{L^p(B(x,r))} \le Cr\|Du\|_{L^p(B(x,r))}$$

for each ball $B(x,r) \subset \mathbb{R}^n$ and each function $u \in W^{1,p}(B^0(x,r))$.

Proof. The estimate of $u \in W^{1,p}(B^0(0,1))$ is a special case of Theorem 4.19, where $U = B^0(0,1)$. Generally, if $u \in W^{1,p}(B^0(x,r))$, let $v(z) = u(x+rz)$. Then $v \in W^{1,p}(B^0(0,1))$, and

$$\|v - (v)_{0,1}\|_{L^p(B(0,1))} \le C\|Dv\|_{L^p(B(0,1))}.$$

The desired result follows from changing variables. $\square$

Space of bounded mean oscillation. A function $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be *of bounded mean oscillation* if

$$\sup_{B(x,r)\subset\mathbb{R}^n} \frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y) - (u)_{x,r}|\, dy < \infty. \tag{4.15}$$

The space of all such functions is called the space of functions of bounded mean oscillation, after dividing out constant functions:

$$\mathrm{BMO}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)/\{\text{constant functions}\},$$

and the left-hand side of (4.15) defines a norm $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}$ on this subspace.

Remark. Let $u \in W^{1,n}(\mathbb{R}^n)$, and $B(x,r) \in \mathbb{R}^n$. By Hölder's and Poincaré's inequalities,

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y) - (u)_{x,r}|\, dy &\le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y) - (u)_{x,r}|^n\, dy\right)^{1/n} \\ &\le \frac{Cr}{|B(x,r)|}\|Du\|_{L^n(B(x,r))} = \frac{C}{|B(0,1)|}\|Du\|_{L^n(B(x,r))}.\end{aligned}$$

Therefore, $W^{1,n}(\mathbb{R}^n)$ is continuously embedded into $\mathrm{BMO}(\mathbb{R}^n)$, and

$$\|u\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C\|Du\|_{L^n(\mathbb{R}^n)} \le C\|u\|_{W^{1,n}(\mathbb{R}^n)}.$$

5 Second-order Elliptic Equations

In this chapter, we study the second-order elliptic equations. The problem we are mostly interested in is the following boundary value problem, which consists of a partial differential equation (PDE) and a homogeneous Dirichlet boundary condition (BC):

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{5.1}$$

where U is a bounded, open subset of $\mathbb{R}^n$, $f : U \to \mathbb{R}$ is a known function, and $u : \overline{U} \to \mathbb{R}$ is the unknown. The partial differential operator L is of second order. Given coefficient functions $a^{ij}, b^i, c,\ (i, j = 1, \cdots, n)$, the operator L is given by the either of the following forms:

- *Divergence form.*

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u. \tag{5.2}$$

- *Non-divergence form.*

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u. \tag{5.3}$$

When the quadratic coefficients $a^{ij} \in C^1(U)$, any of the two forms of L can be rewritten in the other using product rule. For example, the divergence form (5.2) can be written in the non-divergence form:

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} \left(b^i(x) - \sum_{j=1}^{n} a^{ij}_{x_j}\right) u_{x_i} + c(x)u.$$

Both the two forms are discussed in our study, based on the situation.

5.1 The Dual Space of H_0^1

Let U be an open subset of $\mathbb{R}^n$. The Sobolev space $H^1(U) = W^{1,2}(U)$ is a Hilbert space with inner product

$$\langle u, v \rangle_{H^1(U)} = \int_U (uv + Du \cdot Dv)\, dx, \quad u, v \in H^1(U).$$

The space $H_0^1(U)$ is the closure of $C_c^\infty(U)$ in $H^1(U)$. Since $H_0^1(U)$ is a closed subspace of $H^1(U)$, it is also a Hilbert space with the inner product inherited from $H^1(U)$. We write $H^{-1}(U)$ for the dual space to $H_0(U)$:

$$H^{-1}(U) = \left\{ f \mid f : H_0^1(U) \to \mathbb{R} \text{ is a bounded linear functional} \right\}.$$

We write $\langle f, u \rangle$ for the pairing $f(u)$ between $H^{-1}(U)$ and $H_0^1(U)$. If $f \in H^{-1}(U)$, we define it norm

$$\|f\|_{H^{-1}(U)} = \sup\left\{ \langle f, u \rangle : u \in H_0^1(U), \|u\|_{H^{0,1}(U)} \le 1 \right\}$$

By Riesz representation theorem, we have the isomorphism $H^{-1}(U) \cong H_0^1(U)$. However, in this section, we prefer not to identify the space $H_0^1(U)$ with its dual. We point out that, despite the isomorphism, $H^{-1}(U)$ and $H_0^1(U)$ are not equal sets. For further discussion, we study an identification of $H^{-1}(U)$ under the usual L^2 inner product. This characterization of $H^{-1}(U)$ will be useful in the study of second-order linear PDEs.

Theorem 5.1. *Assume $f \in H^{-1}(U)$. Then there exist functions $f^0, f^1, \cdots, f^n \in L^2(U)$ such that*

$$\langle f, v\rangle = \int_U \left(f^0 v + \sum_{i=1}^n f^i v_{x_i} \right) dx, \quad \forall v \in H_0^1(U). \tag{5.4}$$

Furthermore,

$$\|f\|_{H^{-1}(U)} = \inf \left\{ \left(\int_U \sum_{i=0}^n |f^i|^2 \, dx \right)^{1/2} : f^0, f^1, \cdots, f^n \in L^2(U) \text{ satisfies (5.4)} \right\} \tag{5.5}$$

Proof. By Riesz representation theorem, for each $f \in H^{-1}(U)$, there exists $u \in H_0^1(U)$ such that

$$\langle f, v\rangle = \langle u, v\rangle_{H_0^1(U)} = \int_U (uv + Du \cdot Dv) \, dx, \quad \forall v \in H_0^1(U). \tag{5.6}$$

We choose $f^0 = u$, and $f^i = u_{x_i}$ for $i = 1, \cdots, n$. Then we establish (5.4). To show (5.5), assume

$$\langle f, v\rangle = \int_U \left(g^0 v + \sum_{i=1}^n g^i v_{x_i} \right) dx, \quad \forall v \in H_0^1(U)$$

for some $g^0, g^1, \cdots, g^n \in L^2(U)$. Setting $v = u$ in (5.6), we get, by Cauchy's inequality,

$$\int_U (|u|^2 + |Du|^2) \, dx = \int_U \left(g^0 u + \sum_{i=1}^n g^i u_{x_i} \right) dx \le \left(\int_U \sum_{i=0}^n |g^i|^2 \, dx \right)^{1/2} \left(\int_U (|u|^2 + |Du|^2) \, dx \right)^{1/2}.$$

Hence

$$\int_U (|u|^2 + |Du|^2) \, dx = \int_U \sum_{i=0}^n |f^i|^2 \, dx \le \int_U \sum_{i=0}^n |g^i|^2 \, dx. \tag{5.7}$$

Finally, note that when $\|v\|_{H_0^1(U)} \le 1$,

$$\langle f, v\rangle \le \left(\int_U \sum_{i=0}^n |f^i|^2 \, dx \right)^{1/2},$$

and the equality holds when we choose $v = \frac{u}{\|u\|_{H_0^1(U)}}$. Hence

$$\|f\|_{H^{-1}(U)} = \sup \left\{ \langle f \mid v\rangle : v \in H_0^1(U), \|v\|_{H^{0,1}(U)} \le 1 \right\} = \int_U \sum_{i=0}^n |f^i|^2 \, dx. \tag{5.8}$$

Then (5.5) follows from (5.7) and (5.8). □

Remark. (i) Using integration by parts, we can write (5.4) to

$$\langle f, v\rangle = \int_U \left(f^0 - \sum_{i=1}^n f^i_{x_i} \right) v \, dx.$$

Hence we write $f = f^0 - \sum_{i=1}^n f^i_{x_i}$ whenever (5.4) holds.

Also, we obtain a characterization of $H^{-1}(U)$: if $f \in H^{-1}(U)$, then f is the sum of a L^2 function f^0 and the divergence of a vector $(f^1, \cdots, f^n)$ of L^2 functions (in weak/distributional sense).

(ii) If $f \in L^2(U)$, we let $f^0 = f$ and $f^1, \cdots, f^n = 0$. Then $f = f^0 - \sum_{i=1}^n f^i_{x_i} \in H^{-1}(U)$, with

$$\langle f, v\rangle = \langle f, v\rangle_{L^2(U)}.$$

By (5.5), we have $\|f\|_{H^{-1}(U)} \leq \left(\int_U |f^0|^2\,dx\right)^{1/2} \leq \|f\|_{L^2(U)}$. Hence we get the inclusion

$$H^1_0(U) \subset L^2(U) \hookrightarrow H^{-1}(U).$$

We have the following density argument.

Theorem 5.2. *The space $L^2(U)$ is dense in $H^{-1}(U)$.*

Proof. Fix $f \in H^{-1}(U)$. By Riesz representation theorem, we can find $u \in H^1_0(U)$ with $\langle f, v\rangle = \langle u, v\rangle_{H^1_0(U)}$ for all $v \in H^1_0(U)$. We then find an approximation $C^\infty_c(U) \ni u_n \to u$ in $H^1(U)$. Then

$$\langle u_n, v\rangle_{H^1_0(U)} = \int_U (u_n v + Du_n \cdot Dv)\,dx = \int_U (u_n - \Delta u_n) v\,dx. \qquad \text{(integration by parts)}$$

Since $u_n \in C^\infty_c(U)$, we have $u_n - \Delta u_n \in L^2(U)$. Let $f_n : H^1_0(U) \to \mathbb{R}$ be the functional

$$\langle f_n, v\rangle = \langle u_n, v\rangle_{H^1_0(U)} = \langle u_n - \Delta u_n, v\rangle_{L^2(U)}.$$

Then f_n is a bounded linear functional on $L^2(U)$, and

$$|\langle f - f_n, v\rangle| = |\langle u - u_n, v\rangle_{H^1_0(U)}| \leq \|u - u_n\|_{H^1_0(U)}\|v\|_{H^1_0(U)}$$

By taking a supremum on both sides over $\|v\|_{H^1_0(U)} \leq 1$, we obtain $\|f - f_n\|_{H^{-1}(U)} \leq \|u - u_n\|_{H^1_0(U)}$, which converges to 0 as n goes to infinity. Then we complete the proof. $\square$

Remark. In the preceding proof, we identify the space $L^2(U)$ with its dual. In fact, we prove that $(L^2(U))^*$ is dense in the space $H^{-1}(U)$.

5.2 The Lax-Milgram Theorem

In this section, we introduce a general result in Hilbert spaces. We will make use of this result when we establish the weak formulation of PDEs.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H = \sqrt{\langle\cdot,\cdot\rangle_H}$. We continue to write $\langle\cdot,\cdot\rangle$ for the action of an element of H^* on an element of H.

Theorem 5.3 (Lax-Milgram Theorem)**.** *Suppose that $B : H \times H \to \mathbb{R}$ is a bilinear form, for which there exists constants $\alpha, \beta > 0$ such that*

(i) (Boundedness) $|B(u,v)| \leq \alpha\|u\|_H\|v\|_H$ for all $u, v \in H$; and

(ii) (Coercivity) $B(u,u) \geq \beta\|u\|_H^2$ for all $u \in H$.

Then for each $f \in H^*$, there exists a unique $u \in H$ such that

$$B(u,v) = \langle f, v\rangle$$

for all $v \in H$.

Remark. If B is symmetric, i.e. $B(u,v) = B(v,u)$ for all $u, v \in H$, then B becomes a inner product on H, and our result is the Riesz representation theorem.

Proof of Theorem 5.3. We fix $u \in U$, so $B(u,\cdot)$ is a bounded linear functional on H. By Riesz representation theorem, there exists a unique $w_u \in H$ such that $B(u,v) = \langle w_u, v\rangle_H$ for all $v \in H$. We then let $A : H \to H$ be the operator that maps each $u \in H$ to this unique w_u, i.e. $B(u,v) = \langle Au, v\rangle_H$ for all $v \in H$.

- *Claim I. $A \in H^*$.*
  Let $\alpha, \beta \in \mathbb{R}$ and $u_1, u_2 \in H$. Then

$$\begin{aligned}\langle A(\alpha u_1 + \beta u_2), v\rangle_H &= B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v) \\ &= \alpha\langle Au_1, v\rangle_H + \beta\langle Au_2, v\rangle_H = \langle \alpha Au_1 + \beta Au_2, v\rangle_H, \quad \forall v \in H.\end{aligned}$$

  Hence $A(\alpha u_1 + \beta u_2) = \alpha Au_1 + \beta Au_2$, and the linearity follows. To show that A is bounded, note that

$$\|Au\|_H^2 = B(u, Au) \le \alpha\|u\|_H\|Au\|_H \quad \Rightarrow \quad \|Au\|_H \le \alpha\|u\|_H, \ \ \forall u \in H.$$

- *Claim II. A is injective, and the range $\mathfrak{R}(A)$ of A is closed in H.*
  We first show that A is injective. By coercivity,

$$Au = 0 \quad \Rightarrow \quad \|u\|_H^2 \le \frac{1}{\beta}B(u,u) = \langle Au, u\rangle_H = 0 \quad \Rightarrow \quad u = 0 \quad \Rightarrow \quad \ker A = 0.$$

  Next we show that $\mathfrak{R}(A)$ is closed in H. Let $w \in \overline{\mathfrak{R}(A)}$. Then we can find a sequence $w_n \in \mathfrak{R}(A)$ such that $\|w_n - w\|_H \to 0$. Let $u_n = A^{-1}w_n$. By coercivity,

$$\begin{aligned}\|u_n - u_m\|_H \le \frac{B(u_n - u_m, u_n - u_m)}{\beta\|u_n - u_m\|_H} &= \frac{\langle Au_n - Au_m, u_n - u_m\rangle_H}{\beta\|u_n - u_m\|_H} \\ &= \frac{\langle w_n - w_m, u_n - u_m\rangle_H}{\beta\|u_n - u_m\|_H} \le \frac{1}{\beta}\|w_n - w_m\|_H.\end{aligned}$$

  Hence (u_n) is a Cauchy sequence in H. By completeness, we can find $u \in H$ with $\|u_n - u\| \to 0$. Then

$$\|Au - w\|_H \le \|Au - Au_n\|_H + \|Au_n - w\|_H \le \alpha\|u - u_n\|_H + \|w_n - w\|_H \to 0.$$

  Hence $w = Au \in \mathfrak{R}(A)$. Therefore $\mathfrak{R}(A)$ is closed in H.
- *Claim III. $\mathfrak{R}(A) = H$.*
  Since $\mathfrak{R}(A)$ is closed, every $u \in H$ can be uniquely decomposed to $u = u_0 + u_1$ with $u_0 \in \mathfrak{R}(A)$ and $u_1 \in \mathfrak{R}(A)^\perp$. If $\mathfrak{R}(A) \ne H$, we choose $v \in H\backslash\mathfrak{R}(A)$ with orthogonal decomposition $v = v_0 + v_1$. Then for all $u \in H$, we have $\langle Au, v_1\rangle_H = 0$. Setting $u = v_1$, we get $B(v_1, v_1) = \langle Av_1, v_1\rangle_H = 0$, and $v_1 = 0$ by coercivity. This implies $v = v_0 \in \mathfrak{R}(H)$, a contradiction! Therefore $\mathfrak{R}(A) = H$.

Now, combining our *Claims I, II and III*, we conclude that $A : H \to H$ is a bounded linear bijection. By *Banach bounded inverse theorem*, there exists a bounded linear operator $A^{-1} : H \to H$ such that $AA^{-1} = A^{-1}A = \mathrm{Id}$. Then for each $f \in H^*$, by Riesz representation theorem, there exists $w \in H$ such that $\langle f, v\rangle = \langle w, v\rangle_H$ for all $v \in H$. Let $u = A^{-1}w$, then

$$B(u,v) = \langle Au, v\rangle = \langle AA^{-1}w, v\rangle = \langle w, v\rangle = \langle f, v\rangle.$$

Finally, to prove uniqueness, assume $B(u,v) = B(u',v) = \langle f, v\rangle$ for all $v \in H$. By coercivity,

$$\|u - u'\|_H^2 \le \frac{1}{\beta}B(u - u', u - u') = \frac{\langle f, u - u'\rangle - \langle f, u - u'\rangle}{\beta} = 0.$$

Then we complete the proof. $\square$

5.3 Weak Formulation and Poisson's Equation

In this section, we study the weak formulation of the boundary value problem (5.1). Through our discussion, we assume the differential operator is given by the divergence form (5.2):

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u.$$

In fact, the exact solution of a second-order PDE can be intractable. To simplify our problem, we may concern if our PDE holds in the sense of integration, which gives rise to the weak formulation of PDE.

Motivation. We assume that u is a smooth solution of the BVP (5.1). We the multiply the PDE $Lu = f$ by a test function $v \in C_c^\infty(U)$ and integrate over U:

$$\int_U \left(\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right) dx = \int_U fv\, dx.$$

Here we use integration by parts in the first term on the left side, where the boundary term vanishes since $v = 0$ on ∂U. By approximation, we can obtain the same identity when the smooth function v is replaced by $v \in H_0^1(U)$, and the resulting identity make sense if and only if $u \in H_0^1(U)$. Here we incorporate the Dirichlet BCs $u = 0$ on ∂U by choosing $u \in H_0^1(U)$. We require the above identity holds for a weak solution u.

Definition 5.4. The bilinear form $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ associated with the divergence form operator L defined by (5.2) is given by

$$B(u,v) = \int_U \left(\sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right) dx, \quad u, v \in H_0^1(U).$$

When $f \in L^2(U)$, our goal becomes finding a function $u \in H_0^1(U)$ such that $B(u,v) = \langle f, v\rangle_{L^2(U)}$ holds for all $v \in H_0^1(U)$. More generally, we consider the following problem:

$$\begin{cases} Lu = f^0 - \sum_{i=1}^n f^i_{x_i} & \textit{in } U, \\ u = 0 & \textit{on } \partial U, \end{cases} \tag{5.9}$$

where $f = f^0 - \sum_{i=1}^n f^i_{x_i} \in H^{-1}(U)$, and $f_0, f_1, \cdots, f_n \in L^2(U)$.

Definition 5.5 (Weak solutions)**.** Let L be a divergence form operator defined by (5.2), and let B be the associated bilinear form.

(i) Let $f \in L^2(U)$. A function $u \in H_0^1(U)$ is said to be a weak solution to problem (5.1), if

$$B(u,v) = \langle f, v\rangle_{L^2(U)}$$

for all $v \in H_0^1(U)$.

(ii) Let $f = f^0 - \sum_{i=1}^n f^i_{x_i} \in H^{-1}(U)$, and $f_0, f_1, \cdots, f_n \in L^2(U)$. A function $u \in H_0^1(U)$ is said to be a *weak solution* to problem (5.9), if

$$B(u,v) = \langle f, v\rangle$$

for all $v \in H_0^1(U)$, where $\langle f, v\rangle = \int_U (f^0 v + \sum_{i=1}^n f^i v_{x_i})\, dx$ is the pairing of $H^{-1}(U)$ and $H_0^1(U)$.

Example 5.6 (Poisson's equation)**.** Let $f \in H^{-1}(U)$. We consider the following boundary value problem:

$$\begin{cases} -\Delta u = f & \textit{in } U, \\ u = 0 & \textit{on } \partial U. \end{cases}$$

For a divergence form operator L, this is the case $a^{ij}(x) = \delta_{ij}, b^i(x) = 0$ and $c(x) = 0$. The bilinear form associated with the negative Laplacian operator $L = -\Delta$ is given by

$$B(u,v) = \int_U \delta_{ij} u_{x_i} v_{x_j} \, dx = \int_U Du \cdot Dv \, dx,$$

and the weak formulation of this problem is

$$B(u,v) = \langle f, v \rangle, \quad \forall v \in H_0^1(U).$$

Now we study the property of bilinear form B. For any $u, v \in H_0^1(U)$, one can show boundedness:

$$|B(u,v)| = \left| \int_U Du \cdot Dv \, dx \right| \leq \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} \leq \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}.$$

Furthermore, by classical Poincaré's inequality [Corollary 4.6], there exists a constant $C > 0$ such that

$$|B(u,u)| = \int_U |Du|^2 \, dx = \|Du\|_{L^2(U)}^2 \geq \frac{1}{C^2} \|u\|_{L^2(U)}^2, \quad \forall u \in H_0^1(U).$$

Then one can show coercivity:

$$|B(u,u)| = \frac{C^2}{1+C^2} \|Du\|_{L^2(U)}^2 + \frac{1}{1+C^2} \|Du\|_{L^2(U)}^2 \geq \frac{1}{1+C^2} \left(\|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \right) \geq \frac{1}{1+C^2} \|u\|_{H_0^1(U)}^2.$$

Therefore, by Lax-Milgram theorem [Theorem 5.3], *there exists a unique weak solution* $u \in H_0^1(U)$ *to the Poisson's equation under homogeneous Dirichlet boundary conditions.*

Finally, we introduce the definition of elliptic PDEs, which is a generalization of Poisson's equation.

Definition 5.7 (Uniformly elliptic operators)**.** Let L be a partial differential operator of either divergence form (5.2) or non-divergence form (5.3). Assume the coefficient functions $a^{ij}, b^i, c \in L^\infty(U)$ for all $i, j = 1, \cdots, n$, and also assume the symmetry condition

$$a^{ij} = a^{ji}, \quad i, j = 1, \cdots, n.$$

The operator L is said to be *(uniformly) elliptic*, if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

for a.e. $x \in U$ and all $\xi \in \mathbb{R}^n$.

Remark. For each $x \in U$, we write $A(x) = (a^{ij}(x))_{i,j=1}^n$ to be the symmetric $n \times n$ matrix associated with the quadratic coefficients. Ellipticity essentially requires that for a.e. $x \in U$, the matrix $A(x)$ is positive definite, and the smallest eigenvalue is lower bounded by some $\theta > 0$.

5.4 Existence of Weak Solutions

In this section, we discuss the existence of weak solutions for the uniformly elliptic PDE (5.1). Through our discussion, we assume the differential operator is given by the divergence form (5.2):

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u.$$

Recall that a weak solution satisfies the PDE in the sense of integration.

5.4.1 Energy Estimate

The energy estimate focuses on verifying the hypotheses of Lax-Milgram theorem.

Theorem 5.8 (Energy estimates)**.** *Let L be an elliptic partial differential operator, and let B be the associated bilinear form. Then there exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that*

$$|B(u,v)| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}, \tag{5.10}$$

and

$$\beta \|u\|_{H_0^1(U)}^2 \leq B(u,u) + \gamma \|u\|_{L^2(U)}^2 \tag{5.11}$$

for all $u, v \in H_0^1(U)$.

Proof. For all $u, v \in H_0^1(U)$, we can check

$$\begin{aligned}
|B(u,v)| &= \left| \int_U \left(\sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right) dx \right| \\
&\leq \sum_{i,j=1}^{n} \|a^{ij}\|_{L^\infty(U)} \int_U |Du|\,|Dv|\,dx + \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} \int_U |Du|\,|v|\,dx + \|c\|_{L^\infty(U)} \int_U |u|\,|v|\,dx \\
&\leq \max\left\{ \sum_{i,j=1}^{n} \|a^{ij}\|_{L^\infty(U)} + \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)}, \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} \right\} \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}.
\end{aligned}$$

We let

$$\alpha = \max\left\{ \sum_{i,j=1}^{n} \|a^{ij}\|_{L^\infty(U)} + \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)}, \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} + \|c\|_{L^\infty(U)} \right\}.$$

Next, by ellipticity, there exists $\theta > 0$ such that

$$\begin{aligned}
\theta \int_U |Du|^2\,dx &\leq \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j}\,dx = B(u,u) - \int_U \sum_{i=1}^{n} \left(b^i u_{x_i} u + cu^2\right)\,dx \\
&\leq B(u,u) + \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} \int_U |Du|\,|u|\,\,dx + \|c\|_{L^\infty(U)} \int_U |u|^2\,dx \\
&\leq B(u,u) + \epsilon \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} \int_U |Du|^2\,dx + \left(\|c\|_{L^\infty(U)} + \frac{1}{4\epsilon} \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} \right) \int_U |u|^2\,dx,
\end{aligned}$$

where $\epsilon > 0$ is to be chosen. We take $\epsilon > 0$ to be so small that

$$\epsilon \sum_{i=1}^{n} \|b^i\|_{L^\infty(U)} \leq \frac{\theta}{2}.$$

Then for some appropriate constant γ, we have

$$\frac{\theta}{2} \int_U |Du|^2 \, dx \leq B(u,u) + \gamma \int_U |u|^2 \, dx.$$

By classical Poincaré's inequality [Corollary 4.6], there exists a constant $C > 0$ such that

$$\int_U |Du|^2 \, dx \geq \frac{1}{C^2} \int_U |u|^2 \, dx, \quad \forall u \in H_0^1(U).$$

Combining the last two display, we have

$$\int_U |Du|^2 \, dx \geq \frac{1}{1+C^2} \left(\int_U |u|^2 \, dx + \int_U |Du|^2 \, dx \right) \geq \frac{1}{1+C^2} \|u\|_{H_0^1(U)}^2.$$

By setting $\beta = \frac{\theta}{2(1+C^2)}$, we have

$$\beta \|u\|_{H_0^1(U)}^2 \leq B(u,u) + \gamma \|u\|_{L^2(U)}.$$

Thus we complete the proof. $\square$

When $\gamma > 0$ in the energy estimate, the coercivity condition of the Lax-Milgram theorem is not satisfied. The following existence theorem must confront this possibility.

Theorem 5.9 (First existence theorem for weak solutions)**.** *Let L be an elliptic partial differential operator. There is a constant $\gamma \geq 0$ such that for all $\lambda \geq \gamma$ and each function $f = f^0 - \sum_{i=1}^n f_{x_i}^i \in H^{-1}(U)$, where $f^0, f^1, \cdots, f^n \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ to the boundary value problem*

$$\begin{cases} Lu + \lambda u = f^0 - \sum_{i=1}^n f_{x_i}^i & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{5.12}$$

Proof. We consider the operator $L_\lambda = L + \lambda \, \mathrm{Id}$, which has the associated bilinear form

$$B_\lambda(u,v) = B(u,v) + \lambda \langle u, v \rangle_{L^2(U)}, \quad u, v \in H_0^1(U).$$

Take $\gamma \geq 0$ from Theorem 5.8, then B_λ satisfies the hypotheses of Lax-Milgram theorem for all $\lambda \geq \mu$. We then fix $f = f^0 - \sum_{i=1}^n f_{x_i}^i \in L^2(U)$. By Lax-Milgram theorem, there exists a unique $u \in H_0^1(U)$ such that

$$B_\lambda(u,v) = \langle f, v \rangle = \int_U \left(f^0 u + \sum_{i=1}^n f^i u_{x_i} \right) dx.$$

for all $v \in H_0^1(U)$. In fact, u is the unique weak solution of (5.12). $\square$

Remark. In fact, we show that $L_\lambda = L + \lambda \, \mathrm{Id} : H_0^1(U) \to H^{-1}(U)$ is an isomorphism for all $\lambda \geq \gamma$.

5.4.2 The Fredholm Alternative

To study the solvability of elliptic PDEs, we need the tool of Fredholm alternative, which incorporates existence and uniqueness of solutions. To start with, we consider a bounded linear operator T on a Hilbert space H. We have some standard results in functional analysis, about the kernel and range of T and its adjoint:

$$\ker(T) = \mathfrak{R}(T^*)^\perp, \quad \ker(T^*) = \mathfrak{R}(T)^\perp, \quad \overline{\mathfrak{R}(T)} = \ker(T^*)^\perp, \quad \overline{\mathfrak{R}(T^*)} = \ker(T)^\perp.$$

Next, we consider a compact operator $K : H \to H$, i.e. K maps each bounded subset of H to a precompact subset of H. The compactness implies a lot of good properties. Here are some helpful facts:

(i) The adjoint K^* of K is also a compact operator.

(ii) Every nonzero point of $\sigma(K)$ (the spectrum of K) is an eigenvalue of K. In other words, if $\lambda \neq 0$ and $\lambda I - K$ is not invertible, then there exists $x \in H$ such that $Kx = \lambda x$. This implies

$$\ker(\lambda\,\mathrm{Id} - K) = \{0\} \overset{\lambda \neq 0}{\Leftrightarrow} \mathfrak{R}(\lambda\,\mathrm{Id} - K) = H.$$

In other words, when $\lambda \neq 0$, $\lambda\,\mathrm{Id} - K$ is injective if and only if it is surjective.

(iii) If $\lambda \neq 0$, then $\mathfrak{R}(\lambda\,\mathrm{Id} - K)$ is a closed subspace of H.

(iv) If $\lambda \in \sigma(K)\backslash\{0\}$, the eigenspace of K associated with λ is finite dimensional, and

$$\dim\ker(\lambda\,\mathrm{Id} - K) = \dim\ker(\lambda\,\mathrm{Id} - K^*).$$

Therefore, if $K : H \to H$ is a compact operator and $\lambda \neq 0$, the following statements are equivalent:

(a) $\ker(\lambda\,\mathrm{Id} - K) = \{0\}$; (b) $\mathfrak{R}(\lambda\,\mathrm{Id} - K) = H$; (c) $\ker(\lambda\,\mathrm{Id} - K^*) = \{0\}$; (d) $\mathfrak{R}(\lambda\,\mathrm{Id} - K^*) = H$.

We formally summarize our result below.

Theorem 5.10 (Fredholm alternative). *Let K be a compact operator on a Hilbert space H, and fix $\lambda \neq 0$. Then exactly one of the following statements holds:*

(a) For every $v \in H$, the equation $\lambda u - Ku = v$ has a unique solution $u \in H$;

(b) The eigenvalue problem $Ku = \lambda u$ has nonzero solution $u \neq 0$ in H.

Furthermore, if (a) holds for K, it also holds for the adjoint operator K^*; otherwise, (b) holds for both the operator K and its adjoint operator K^*, and their eigenspaces associated with λ has the same dimension.

Remark. We can interpret the basic results as follows: In an appropriately formulated problem, either

(a) The inhomogeneous equation can be solved uniquely for each choice of data, or

(b) The homogeneous equation has a nontrivial solution.

Adjoint operators. We assume that $b^i \in C^1(\overline{U})$. If $u, v \in H_0^1(U)$, we use integration by parts to obtain

$$\begin{aligned}
\int_U (Lu)v\,dx &= \int_U \left(-\sum_{i,j=1}^n \left(a^{ij}u_{x_i}\right)_{x_j} + \sum_{i=1}^n b^i u_{x_i} + cu \right) v\,dx = \int_U \left(\sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} - \sum_{i=1}^n (b^i v)_{x_i} u + cuv \right) dx \\
&= \int_U \sum_{i,j=1}^n \left(a^{ij}u_{x_i}v_{x_j} - \sum_{i=1}^n b^i u v_{x_i} + \left(c - \sum_{i=1}^n b_{x_i} \right) uv \right) dx \\
&= \int_U u \left(-\sum_{i,j=1}^n (a^{ij}v_{x_j})_{x_i} - \sum_{i=1}^n b^i v_{x_i} + \left(c - \sum_{i=1}^n b_{x_i} \right) v \right) dx.
\end{aligned}$$

This identity has a form similar to the definition of adjoint: $\langle Lu, v\rangle_{L^2(U)} = \langle u, L^*v\rangle_{L^2(U)}$.

Definition 5.11 (Adjoint)**.** Let L be an divergence form elliptic operator with $b_i \in C^1(\overline{U})$ for all $i = 1, \cdots, n$. The operator L^*, called the *formal adjoint* of L, is defined as

$$L^*v = -\sum_{i,j=1}^{n} (a^{ij} v_{x_j})_{x_i} - \sum_{i=1}^{n} b^i v_{x_i} + \left(c - \sum_{i=1}^{n} b_{x_i} \right) v.$$

The adjoint bilinear form $B^* : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$, associated with L^*, is defined by

$$B^*(v, u) = B(u, v), \quad u, v \in H_0^1(U).$$

Fix $f \in H^{-1}(U)$. We say that $v \in H_0^1(U)$ is a weak solution of the adjoint problem

$$\begin{cases} L^*v = f & \textit{in } U, \\ v = 0 & \textit{on } \partial U, \end{cases}$$

if $B^*(v, u) = \langle f, u \rangle$ for all $u \in H_0^1(U)$, where $\langle \cdot, \cdot \rangle$ is the pairing between $H^{-1}(U)$ and $H_0^1(U)$.

We derive an existence theorem for weak solutions of elliptic PDEs using Fredholm alternative.

Theorem 5.12 (Second existence theorem for weak solutions)**.** *Let L be a elliptic operator.*

(i) Exactly one of the following statements holds: either

(a) for each $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ of the boundary value problem

$$\begin{cases} Lu = f & \textit{in } U, \\ u = 0 & \textit{on } \partial U, \end{cases} \tag{5.13}$$

or else

(b) there exists a nonzero weak solution $u \neq 0$ in $H_0^1(U)$ of the homogeneous problem

$$\begin{cases} Lu = 0 & \textit{in } U, \\ u = 0 & \textit{on } \partial U. \end{cases} \tag{5.14}$$

The dichotomy (a) & (b) is the Fredholm alternative.

(ii) Furthermore, should (b) hold, the dimension of the subspace $N \subset H_0^1(U)$ of weak solutions of (5.14) is finite and equals the dimension of the subspace $N^* \in H_0^1(U)$ of weak solutions of the adjoint problem

$$\begin{cases} L^*v = 0 & \textit{in } U, \\ v = 0 & \textit{on } \partial U. \end{cases} \tag{5.15}$$

(iii) Finally, the boundary value problem (5.13) has a weak solution if and only if

$$\langle f, v \rangle_{L^2(U)} = 0, \quad \forall v \in N^*.$$

Proof. Step I. We choose $\lambda = \gamma$ in Theorem 5.9, and assume without loss of generality $\gamma > 0$. Let

$$B_\gamma(u, v) = B(u, v) + \gamma \langle u, v \rangle_{L^2(U)}, \quad u, v \in H_0^1(U),$$

which is the bilinear form associated with the operator $L_\gamma = L + \gamma\,\mathrm{Id}$. Then for each $g \in L^2(U)$ there exists a unique $u \in H_0^1(U)$ solving $B_\gamma(u, v) = \langle g, v \rangle_{L^2(U)}$ for all $v \in H_0^1(U)$. We define the inverse $L_\gamma^{-1} : L^2(U) \to$

$H_0^1(U)$ by writing $u = L_\gamma^{-1} g$. For $f \in L^2(U)$, we observe that u is a weak solution of (5.13) if and only if

$$u = L_\gamma^{-1}(\gamma u + f).$$

We let $K = \gamma L_\gamma^{-1}$ and $h = L_\gamma^{-1} f$. Then we rewrite this problem to $(\mathrm{Id} - K)u = h$. To employ Fredholm alternative, we claim that $K : L^2(U) \to L^2(U)$ is a compact bounded linear operator. To this end, we note that, by energy estimate [Theorem 5.8] (5.11), for $g \in L^2(U)$ and $u = L_\gamma^{-1} g$,

$$\beta \|u\|^2_{H_0^1(U)} \le B_\gamma(u,u) = \langle g, u\rangle_{L^2(U)} \le \|g\|_{L^2(U)} \|u\|_{L^2(U)} \le \|g\|_{L^2(U)} \|u\|_{H_0^1(U)}.$$

Then

$$\|Kg\|_{H_0^1(U)} \le \frac{\gamma}{\beta} \|g\|_{L^2(U)}.$$

By Rellich-Kondrachov compactness theorem, we have $H^1(U) \Subset L^2(U)$, hence every bounded subset of $H^1(U)$ is precompact in $L^2(U)$, and $K : L^2(U) \to L^2(U)$ is a compact operator.

Step II. According to the Fredholm alternative, exactly one of the following statements holds: either

(a) For each $h \in L^2(U)$, the equation $(\mathrm{Id} - K)u = h$ has a unique solution $u \in L^2(U)$; or else,

(b) The equation $(\mathrm{Id} - K)u = 0$ has a nonzero solution $u \ne 0$ in $L^2(U)$.

Should the statement (a) holds, we fix any $f \in L^2(U)$, and set $h = L_\gamma^{-1} f \in H_0^1(U) \subset L^2(U)$. Then we find a unique $u \in L^2(U)$ with $(\mathrm{Id} - K)u = h$, and in fact $u = Ku + h \in H_0^1(U)$. This is the weak solution to (5.13).

Should the statement (b) holds, the nonzero solution $u = Ku \in H_0^1(U)$. Furthermore, the space N of solutions of (5.14) is $\ker(\mathrm{Id} - K)$. According to Theorem 5.10, N is of finite dimension. A similar procedure shows that the space N^* of solutions of (5.15) is $\ker(\mathrm{Id} - K^*)$, which has the same dimension as N.

Finally, when the statement (b) holds, the problem (5.13) is has a weak solution if and only if the equation $(\mathrm{Id} - K)u = h$ has a solution, if and only if $h \in \mathfrak{R}(\mathrm{Id} - K) = \ker(\mathrm{Id} - K^*)^\perp = (N^*)^\perp$. Note that for all $v \in N^*$,

$$\langle f, v\rangle_{L^2(U)} = \langle f, K^* v\rangle_{L^2(U)} = \langle Kf, v\rangle_{L^2(U)} = \gamma \langle h, v\rangle_{L^2(U)}.$$

Therefore, the boundary problem (5.13) has a weak solution if and only if $f \in (N^*)^\perp$. □

We also have the following result concerning the solvaibility of problems in the form of (5.12).

Theorem 5.13 (Third existence theorem for weak solutions)**.** *Let L be a elliptic operator.*

(i) There exists an at most countable set $\Sigma \subset \mathbb{R}$ such that the boundary value problem

$$\begin{cases} Lu = \lambda u + f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{5.16}$$

has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$.

(ii) If Σ is infinite, then $\Sigma = \{\lambda_k\}_{k=0}^\infty$, the values of a nondecreasing sequence with $\lambda_k \to \infty$.

Proof. We take the constant γ from Theorem 5.9, and assume without loss of generality $\gamma > 0$. Let $\lambda > -\gamma$. According to Fredholm alternative [Theorem 5.10], the boundary value problem (5.16) has a unique solution for each $f \in L^2(U)$ if and only if 0 is not an eigenvalue of L; that is, $u = 0$ is the only weak solution of the following homogeneous problem:

$$\begin{cases} L_\gamma u = (\gamma + \lambda) u & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

where $L_\gamma = L + \gamma\,\mathrm{Id}$. The PDE holds when

$$u = (\lambda+\gamma)L_\gamma^{-1}u = \frac{\gamma+\lambda}{\gamma}Ku,$$

where $K = \gamma L_\gamma^{-1}$ is a compact bounded linear operator on $L^2(U)$. Therefore, the boundary value problem (5.16) has a unique solution for each $f \in L^2(U)$ if and only if $\frac{\gamma}{\gamma+\lambda}$ is not an eigenvalue of K.

Since K is a compact operator on $L^2(U)$, its spectrum $\sigma(K)$ is either a finite set or the values of a sequence converging to 0. Then the set Σ has at most countably many values, and $\lambda_k \to \infty$ if Σ is infinite. □

Remark. The set Σ is called the *(real) spectrum* of the operator L. When $\lambda \in \Sigma$, by the Fredholm alternative, the following eigenvalue problem has nonzero solution $u \neq 0$ in $H_0^1(U)$:

$$\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Theorem 5.14 (Boundedness of the inverse)**.** *If $\lambda \notin \Sigma$, there exists a constant C such that for all $f \in L^2(U)$,*

$$\|u\|_{L^2(U)} \le C\|f\|_{L^2(U)},$$

where u is the unique weak solution of problem (5.16). The constant C depends only on λ, U and L.

Proof. Argue by contradiction. Assume that there exists sequences $f_k \in L^2(U)$ and $u_k \in H_0^1(U)$ such that u_k is a weak solution of (5.16) when $f = f_k$:

$$\begin{cases} Lu_k = \lambda u_k + f_k & \text{in } U, \\ u_k = 0 & \text{on } \partial U, \end{cases}$$

but $\|u_k\|_{L^2(U)} > k\|f_k\|_{L^2(U)}$, $k = 1, 2, \cdots$. We may also assume with no loss that $\|u_k\|_{L^2(U)} = 1$, so $f_k \to 0$ in $L^2(U)$. According to the energy estimate, the sequence (u_k) is also bounded in $H_0^1(U)$:

$$\begin{aligned} \beta\|u_k\|_{H_0^1(U)} &\le B(u_k, u_k) + \gamma\|u_k\|_{L^2(U)} \\ &= \langle \lambda u_k + f_k, u_k\rangle_{L^2(U)} + \gamma\|u_k\|_{L^2(U)}^2 < \frac{1}{k} + \lambda + \gamma \le 1 + \lambda + \gamma. \end{aligned}$$

By Banach-Alaoglu theorem and Rellich-Kondrachov theorem, there exists a subsequence (u_{k_j}) such that

$$u_{k_j} \to u \text{ weakly in } H_0^1(U), \quad \text{and} \quad u_{k_j} \to u \text{ in } L^2(U).$$

Since $B(\cdot, v)$ is a bounded linear functional on $H_0^1(U)$ for all $v \in H_0^1(U)$, we have

$$B(u,v) = \lim_{j\to\infty} B(u_{k_j}, v) = \lim_{j\to\infty} \langle \lambda u_{k_j} + f_{k_j}, v\rangle_{L^2(U)} = \langle \lambda u, v\rangle_{L^2(U)}.$$

Therefore u is a weak solution of the homogeneous problem

$$\begin{cases} Lu = \lambda u & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases}$$

Since $\lambda \notin \Sigma$, we have $u \equiv 0$ by the Fredholm alternative. However $\|u\|_{L^2(U)} = 1$, because $u_{k_j} \to u$ in $L^2(U)$, leading to a contradiction! □

5.5 Regularity Theory

In this section, we study the smoothness of the weak solution to the second-order elliptic PDE

$$Lu = f \quad in\ U.$$

5.5.1 Difference Quotients

We first study difference quotient approximations to weak derivatives.

Definition 5.15 (Difference quotient)**.** Let $u \in L^1_{\text{loc}}(U)$ and $V \Subset U$. The i^{th} *difference quotient of size* h is

$$D_i^h u(x) = \frac{u(x+he_i) - u(x)}{h}, \quad i = 1, 2, \cdots, n,$$

where $x \in V$ and $0 < |h| < d(V, \partial U)$. The *difference quotient of size* h is $D^h u = (D_1^h u, D_2^h u, \cdots, D_n^h u)$.

Remark. If $\operatorname{supp} v \subset \overline{V}$ and $0 < |h| < \frac{1}{2} d(V, \partial U)$, we have the integration-by-parts formula

$$\int_U v(x) D_i^h u(x)\, dx = -\int_U u(x) D_i^{-h} v(x)\, dx.$$

Also,

$$D_i^h(uv) = u_i^h D_i^h v + v D_i^h u,$$

where $u_i^h(x) = u(x + he_i)$.

Theorem 5.16 (Difference quotients and weak derivatives)**.** *Let $V \Subset U \subset \mathbb{R}^n$, and $u \in L^1_{\text{loc}}(U)$.*

(i) Let $1 \le p < \infty$ and $u \in W^{1,p}(U)$. Then there exists a constant $C > 0$ depending only on p and n such that for all $0 < |h| < \frac{1}{2} d(V, \partial U)$,

$$\|D^h u\|_{L^p(V)} \le C \|Du\|_{L^p(U)}.$$

(ii) Let $1 < p < \infty$ and $u \in L^p(V)$. If there exist constants $C, \epsilon > 0$ such that $\|D^h u\|_{L^p(V)} \le C$ for all $0 < |h| < \epsilon$, then $u \in W^{1,p}(V)$, and $\|Du\|_{L^p(V)} \le C$.

Proof. (i) Assume $1 \le p < \infty$ and u is smooth. If $x \in V$ and $0 < h < \frac{1}{2} d(V, \partial U)$,

$$D_i^h u(x) = \frac{u(x+he_i) - u(x)}{h} = \frac{1}{h} \int_0^h u_{x_i}(x + te_i)\, dt.$$

We may assume $h > 0$, and the case $h < 0$ is similar. By Holder's inequality,

$$\left|D_i^h u(x)\right| \le \frac{1}{h} \int_0^h |u_{x_i}(x + te_i)|\, dt \le h^{-1/p} \left(\int_0^h |u_{x_i}(x + te_i)|^p\, dt \right)^{1/p},$$

Then

$$\begin{aligned}
\int_V |D^h u|^p\, dx &\le C \sum_{i=1}^n \int_V \frac{1}{h} \int_0^h |u_{x_i}(x + te_i)|^p\, dt\, dx = \frac{C}{h} \sum_{i=1}^n \int_0^h \int_V |u_{x_i}(x + te_i)|^p\, dx\, dt \\
&\le \frac{C}{h} \sum_{i=1}^n \int_0^h \int_U |u_{x_i}(x)|^p\, dx\, dt = C \|Du\|_{L^p(U)}^p.
\end{aligned}$$

The general statement $u \in W^{1,p}(U)$ follows from the density of smooth functions in $W^{1,p}(U)$.

(ii) Assume that $\|D_i^h u\|_{L^p(V)}$ for all $0<|h|<\epsilon$, and $\phi\in C_c^\infty(V)$. Then

$$\int_V u(x)D_i^h\phi(x)\,dx = -\int_V D_i^{-h}u(x)\phi(x)\,dx.$$

Since $(\|D^h u\|_{L^p(V)})_{0<|h|<\epsilon}$ is bounded, there exists a subsequence $h_k\downarrow 0$ such that $D_i^{h_k}u$ converges weakly in $L^p(V)$ for each $i\in\{1,2,\cdots,n\}$. Let $v_i\in L^p(V)$ be the weak limit. Then

$$\begin{aligned}\int_V u(x)\phi_{x_i}(x) &= \lim_{k\to\infty}\int_V u(x)D_i^{h_k}\phi(x) = -\lim_{k\to\infty}\int_V D_i^{-h_k}u(x)\phi(x)\,dx\\ &= -\int_V v_i(x)\phi(x)\,dx = -\int_U v_i(x)\phi(x)\,dx.\end{aligned}$$

Hence $u_{x_i}=v_i$ in the weak sense, and $Du\in L^p(V,\mathbb{R}^n)$, with $\|Du\|_{L^p(V)}\le C$. ☐

Remark. Variants of this Theorem can hold even if it is not true that $V\Subset U$. For example, if U is the open half ball $B(0,1)\cap\{x_n>0\}$ and $V=B(0,\frac{1}{2})\cap\{x_n>0\}$, we have $\|D_i^h u\|_{L^p(V)}\le\|u_{x_i}\|_{L^p(U)}$ for all $0<|h|<\frac{1}{4}$ and all $i=1,2,\cdots,n-1$.

5.5.2 Interior Regularity

We first study the regularity of the weak solution in the interior of the domain $U\subset\mathbb{R}^n$, and we do not require the boundary condition $u=0$ on ∂U. Recall that L is the differetial operator of the divergence form

$$Lu = -\sum_{i,j=1}^n \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u.$$

Theorem 5.17 (Interior H^2-regularity)**.** *Assume that $a^{ij}\in C^1(U)\cap L^\infty(U)$ and $b^i,c\in L^\infty(U)$ for all $i,j=1,2,\cdots,n$, and $f\in L^2(U)$. If $u\in H^1(U)$ is a weak solution of the elliptic PDE*

$$Lu=f \quad \text{in } U, \tag{5.17}$$

then $u\in H^2_{\mathrm{loc}}(U)$. Furthermore, for each open set $V\Subset U$, there exists a constant C depending on U,V and the coefficients of L such that

$$\|u\|_{H^2(V)}\le C(\|u\|_{L^2(U)}+\|f\|_{L^2(U)}).$$

Proof. We fix an open set $V\Subset U$, and take an open set W with $V\Subset W\Subset U$. By C^∞-Urysohn lemma, we take a smooth function $\zeta:\mathbb{R}^n\to[0,1]$ such that $\zeta=1$ on $\overline{V}$, and $\zeta=0$ on $\mathbb{R}^n\backslash W$.

Step I. Since u is a weak solution of (5.17), we have $B(u,v)=\langle f,v\rangle$ for all $v\in H_0^1(U)$. Then

$$\sum_{i,j=1}^n\int_U a^{ij}u_{x_i}v_{x_j}\,dx = \int_U\left(f-\sum_{i=1}^n b^iu_{x_i}-cu\right)v\,dx. \tag{5.18}$$

We take $|h|>0$ sufficiently small and $k\in\{1,2,\cdots,n\}$, and substitute $v=-D_k^{-h}(\zeta^2D_k^hu)$ into (5.18). We write the resulting equation as $A=B$, where

$$A=-\sum_{i,j=1}^n\int_U a^{ij}u_{x_i}\left[D_k^{-h}(\zeta^2D_k^hu)\right]_{x_j}dx,\quad\text{and}\quad B=-\int\left(f-\sum_{i=1}^n b^iu_{x_i}-cu\right)D_k^{-h}\left(\zeta^2D_k^hu\right)dx.$$

We then estimate the terms A and B.

Step II. For the term A, we have

$$\begin{aligned}
A &= \sum_{i,j=1}^{n} \int_U D_k^h(a^{ij}u_{x_i}) \left(\zeta^2 D_k^h u\right)_{x_j} dx \\
&= \sum_{i,j=1}^{n} \left(\int_U a_k^{ij,h} \left(D_k^h u_{x_i}\right) \left(\zeta^2 D_k^h u\right)_{x_j} dx + \int_U u_{x_i} \left(D_k^h a^{ij}\right) \left(\zeta^2 D_k^h u\right)_{x_j} dx \right) \\
&= \sum_{i,j=1}^{n} \int_U a_k^{ij,h} D_k^h u_{x_i} D_k^h u_{x_j} \zeta^2 \, dx \\
&\quad + \sum_{i,j=1}^{n} \int_U \left[2a_k^{ij,h} D_k^h u_{x_i} D_k^h u \zeta \zeta_{x_j} + u_{x_i} \left(D_k^h a^{ij}\right) D_k^h u_{x_j} \zeta^2 + 2u_{x_i} \left(D_k^h a^{ij}\right) D_k^h u \zeta \zeta_{x_j} \right] dx \\
&=: A_1 + A_2.
\end{aligned}$$

The uniform ellipcity condition implies

$$A_1 \geq \theta \int_U \zeta^2 |D_k^h Du|^2 \, dx.$$

Since $a^{ij} \in C^1(U) \cap L^\infty(U)$, there exists an appropriate constant C_1 depending on (a^{ij}) and ζ such that

$$\begin{aligned}
|A_2| &\leq C_1 \int_U \left(|D_k^h Du| \left|D_k^h u\right| + |D_k^h Du| \, |Du| + |D_k^h u| \, |Du|\right) \zeta \, dx \\
&\leq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \, dx + \left(\frac{C_1^2}{\theta} + C_1 \right) \int_W \left(|D_k^h u|^2 + |Du|^2\right) dx.
\end{aligned}$$

By Theorem 5.16 (i), we have $\|D^h u\|_{L^2(W)} \leq C_2 \|Du\|_{L^2(U)}$ for some constant C_2. Combining the last three displays gives

$$A \geq \frac{\theta}{2} \int_U \zeta^2 |D_k^h Du|^2 \, dx - \left(\frac{C_1^2}{\theta} + C_1 \right) (1 + C_2) \int_U |Du|^2 \, dx. \tag{5.19}$$

Step III. For the term B, we can find a constant C_3 depending on coefficients b^i and c such that

$$|B| \leq C_3 \int_U (|f| + |Du| + |u|) \, |v| \, dx. \tag{5.20}$$

By Theorem 5.16 (i), we can find constants C_4 and C_5 such that

$$\begin{aligned}
\int_U |v|^2 \, dx &\leq C_4 \int_U |D(\zeta^2 D_k^h u)|^2 \, dx \leq 8C_4 \int_W |D_k^h u|^2 \, dx + 2C_4 \int_W \zeta^2 |D_k^h Du|^2 \, dx \\
&\leq C_5 \left(\int_U |Du|^2 \, dx + \int_U \zeta^2 |D_k^h Du|^2 \, dx \right).
\end{aligned}$$

Combining (5.20) and the last display gives

$$|B| \leq \frac{\theta}{4} \int_U \zeta^2 |D_k^h Du|^2 \, dx + \left(\frac{4C_3}{\theta} + C_5 \right) \int_U \left(|f|^2 + |Du|^2 + |u|^2\right) dx. \tag{5.21}$$

Step IV. Since $A = B$, we combine the estimates (5.19) and (5.21) to obtain for all $k = 1, 2, \cdots, n$ and all sufficiently small $|h| > 0$ that

$$\int_V |D_k^h Du|^2 \, dx \leq \int_U \zeta^2 |D_k^h Du|^2 \, dx \leq C_6 \int_U \left(|f|^2 + |Du|^2 + |u|^2\right) dx$$

where C_6 is an appropriate constant. By Theorem 5.16 (ii), $|Du| \in H^1_{\mathrm{loc}}(U;\mathbb{R}^n)$, and

$$\|u\|_{H^2(V)} \le C_7 \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}\right). \tag{5.22}$$

Step V. Since $V \Subset W$, we can take $V \Subset \widetilde{V} \Subset W$. Proceeding exactly as in Steps I-IV with $V, \widetilde{V}, W$ replacing the roles of V, W, U, respectively. Then the estimate (5.22) is refined to

$$\|u\|_{H^2(V)} \le C_8 \left(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}\right), \tag{5.23}$$

where C_8 is an appropriate constant depending on V, W, etc. We take a new smooth function $\eta : \mathbb{R}^n \to [0,1]$ such that $\eta = 1$ on W, and $\eta = 0$ on $\mathbb{R}^n \backslash \widetilde{W}$ for some $W \Subset \widetilde{W} \Subset U$. Then we set $v = \eta^2 u$ in (5.18) to obtain

$$\sum_{i,j=1}^n \int_U a^{ij}\eta^2 u_{x_i}u_{x_j}\,dx + 2\sum_{i,j=1}^n \int_U a^{ij}uu_{x_i}\eta\eta_{x_j}\,dx = \int_U \left(f - \sum_{i=1}^n b^i u_{x_i} - cu\right)\eta^2 u\,dx. \tag{5.24}$$

By uniform ellipcity abd Cauchy-Schwarz iequality, the following estimate holds for the left-hand side of (5.24):

$$\begin{aligned}\sum_{i,j=1}^n \int_U a^{ij}\eta^2 u_{x_i}u_{x_j}\,dx + 2\sum_{i,j=1}^n \int_U a^{ij}uu_{x_i}\eta\eta_{x_j}\,dx &\ge \theta\int_U \eta^2|Du|^2\,dx - 2\sum_{i,j=1}^n \int_U |a^{ij}uu_{x_i}|\cdot\eta\eta_{x_j}\,dx \\ &\ge \theta\|\eta Du\|^2_{L^2(U)} - C_9\|\eta Du\|_{L^2(U)}\|u\|_{L^2(U)}.\end{aligned} \tag{5.25}$$

Also the right-hand side satisfies

$$\int_U \left(f - \sum_{i=1}^n b^i u_{x_i} - cu\right)\eta^2 u\,dx \le C_{10}\|u\|_{L^2(U)}\left(\|f\|_{L^2(U)} + \|\eta Du\|_{L^2(U)} + \|u\|_{L^2(U)}\right). \tag{5.26}$$

Combining (5.24), (5.25) and (5.26), we have

$$\theta\|\eta Du\|^2_{L^2(U)} - (C_9 + C_{10})\|Du\|_{L^2(W)}\|u\|_{L^2(W)} - C_{10}\|u\|_{L^2(W)}\left(\|f\|_{L^2(W)} + \|u\|_{L^2(W)}\right) \le 0,$$

which implies

$$\|\eta Du\|_{L^2(U)} \le \frac{C_9 + C_{10}}{2\theta}\|u\|_{L^2(U)} + \sqrt{\frac{(C_9+C_{10})^2}{4\theta^2}\|u\|^2_{L^2(U)} + \frac{C_{10}}{\theta}\|u\|_{L^2(U)}\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)}.$$

Then

$$\int_W |Du|^2\,dx \le \int_U \eta^2|Du|^2\,dx = \|\eta Du\|_{L^2(U)} \le C_{11}\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

We plug-in this estimate to (5.23) to obtain

$$\|u\|_{H^2(V)} \le C_{12}\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right).$$

Then we finish the proof. $\square$

Remark. The function $u \in H^2_{\mathrm{loc}}(U)$ is called a *strong solution* of (5.17), because u actually solves the PDE. Since $u \in H^2_{\mathrm{loc}}(U)$, the integration-by-parts formula implies

$$\langle Lu, \phi\rangle_{L^2(U)} = B(u,\phi) = \langle f, \phi\rangle_{L^2(U)}, \quad \phi \in C_c^\infty(U).$$

Hence $\langle Lu - f, \phi\rangle_{L^2(U)} = 0$ for all $\phi \in C_c^\infty(U)$, and $Lu = f$ a.e..

Theorem 5.18 (Higer-order regularity)**.** *Let $m \in \mathbb{N}_0$, and assume that $a^{ij}, b^i, c \in C^{m+1}(U) \cap L^\infty(U)$ for all $i, j = 1, 2, \cdots, n$ and $f \in H^m(U)$. If $u \in H^1(U)$ is a weak solution of the elliptic equation*

$$Lu = f \quad \text{in } U, \tag{5.27}$$

then $u \in H^{m+2}_{\text{loc}}(U)$. Futhermore, for each open set $V \Subset U$, there exists a constant C depending on U, V, m and the coefficients of L such that

$$\|u\|_{H^{m+2}(V)} \le C \left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)} \right).$$

Proof. We establish the desired results by induction on m. The result $m = 0$ follows from Theorem 5.17,

Step I. We assume that our statements are valid for some $m \in \mathbb{N}$. If $a^{ij}, b^i, c \in C^{m+2}(U) \cap L^\infty(U)$ for all $i, j = 1, 2, \cdots, n$ and $f \in H^{m+1}(U)$, by the induction hypotheses, if $u \in H^1(U)$ is a weak solution of (5.27), then $u \in H^{m+2}_{\text{loc}}(U)$, and for each $W \Subset U$, there exists a constant $C_1 > 0$ depending on U, W and L such that

$$\|u\|_{H^{m+2}(W)} \le C_1 \left(\|f\|_{H^m(U)} + \|u\|_{L^2(U)} \right). \tag{5.28}$$

Step II. We fix $V \Subset W \Subset U$ and a multi-index α with $|\alpha| = m + 1$. For each $\phi \in C_c^\infty(W)$,

$$\begin{aligned}
B(u, D^\alpha \phi) &= \int_U \left(\sum_{i,j=1}^n a^{ij} u_{x_i} (D^\alpha \phi)_{x_j} + \sum_{i=1}^n b^i u_{x_i} D^\alpha \phi + cu D^\alpha \phi \right) dx \\
&= (-1)^m \int_U \left(\sum_{i,j=1}^n \phi_{x_j} D^\alpha (a^{ij} u_{x_i}) + \sum_{i=1}^n \phi D^\alpha (b^i u_{x_i}) + \phi D^\alpha (cu) \right) dx \\
&= (-1)^{m+1} \int_U \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \left(\sum_{i,j=1}^n (D^{\alpha-\beta} a^{ij} D^\beta u_{x_i}) \phi_{x_j} + \sum_{i=1}^n (D^{\alpha-\beta} b^i D^\beta u_{x_i}) \phi + (D^{\alpha-\beta} c D^\beta u) \phi \right) dx \\
&= (-1)^{m+1} \int_U \sum_{\beta < \alpha} \binom{\alpha}{\beta} \left(- \sum_{i,j=1}^n (D^{\alpha-\beta} a^{ij} D^\beta u_{x_i})_{x_j} + \sum_{i=1}^n (D^{\alpha-\beta} b^i D^\beta u_{x_i}) + (D^{\alpha-\beta} c D^\beta u) \right) \phi \, dx \\
&\quad + (-1)^{m+1} \int_U \left(\sum_{i,j=1}^n a^{ij} (D^\alpha u)_{x_i} \phi_{x_j} + \sum_{i=1}^n b^i (D^\alpha u)_{x_i} \phi + c (D^\alpha u) \phi \right) dx.
\end{aligned}$$

Since u is a weak solution of (5.27), we have $B(u, D^\alpha \phi) = \langle f, D^\alpha \phi \rangle_{L^2(U)} = (-1)^{m+1} \langle D^\alpha f, \phi \rangle_{L^2(U)}$. Let

$$\widetilde{f} = D^\alpha f - \sum_{\beta < \alpha} \binom{\alpha}{\beta} \left(- \sum_{i,j=1}^n (D^{\alpha-\beta} a^{ij} D^\beta u_{x_i})_{x_j} + \sum_{i=1}^n (D^{\alpha-\beta} b^i D^\beta u_{x_i}) + (D^{\alpha-\beta} c D^\beta u) \right). \tag{5.29}$$

Then the last two displays imply

$$B(D^\alpha u, \phi) = \langle \widetilde{f}, \phi \rangle_{L^2(U)},$$

which holds for all $\phi \in C_c^\infty(W)$, and by density for all $\phi \in H_0^1(W)$. Hence $\widetilde{u} = D^\alpha u$ is a weak solution of

$$L\widetilde{u} = \widetilde{f} \quad \text{in } W.$$

By (5.28) and (5.29),

$$\|\widetilde{f}\|_{L^2(W)} \le C_2 \left(\|f\|_{H^{m+1}(U)} + \|u\|_{H^{m+2}(U)} \right) \le C_3 \left(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)} \right)$$

Step III. By Theorem 5.17 and estimates (5.28)-(5.29), we see that $D^\alpha u \in H^2(V)$, and

$$\|D^\alpha u\|_{H^2(V)} \leq C_4 \left(\|\widetilde{f}\|_{L^2(W)} + \|D^\alpha u\|_{L^2(W)}\right) \leq C_5 \left(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}\right).$$

This result is valid for all multi-indices $|\alpha| = m+1$. Hence $u \in H^{m+3}(V)$, and

$$\|u\|_{H^{m+3}(V)} \leq C_6 \left(\|u\|_{H^{m+2}(V)} + \sum_{|\alpha|=m+1} \|D^\alpha u\|_{H^2(V)}\right) \leq C_7 \left(\|f\|_{H^{m+1}(U)} + \|u\|_{L^2(U)}\right).$$

Then we conclude the proof. $\square$

Remark. By Theorem 4.12, if $2(m+2) > n$, we can conclude that the weak solution $u \in C^{m+1-\lfloor \frac{n}{2} \rfloor}(U)$.

Theorem 5.19 (Infinite differentiability in the interior)**.** *Assume that $a^{ij}, b^i, c \in C^\infty(U) \cap L^\infty(U)$ for all $i, j = 1, 2, \cdots, n$ and $f \in C^\infty(U)$. If $u \in H^1(U)$ is a weak solution of the elliptic equation*

$$Lu = f \quad \text{in } U, \tag{5.30}$$

then $u \in C^\infty(U)$.

Proof. By Theorem 5.18, $u \in H^m_{\text{loc}}(U)$ for all integers $m \in \mathbb{N}$. We fix $V \Subset U$. According to Theorem 4.12, $u \in C^k(V)$ for each $k > \frac{n}{2}$ by modifying u on a Lebesgue null set if necessary, and hence $u \in C^\infty(V)$. Since $V \Subset U$ is arbitrary, $u \in C^\infty(U)$. $\square$

5.5.3 Boundary Regularity

Now we study the regularity of the weak solution up to the boundary of the domain $U \subset \mathbb{R}^n$. The function u we study is a weak solution of the BVP

$$\begin{cases} Lu = f & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases}$$

Theorem 5.20 (Boundary H^2-regularity)**.** *Assume that $a^{ij} \in C^1(\overline{U})$ and $b^i, c \in L^\infty(U)$ for all $i, j = 1, 2, \cdots, n$, and $f \in L^2(U)$. Assume further that ∂U is C^2. If $u \in H_0^1(U)$ is a weak solution of the BVP*

$$\begin{cases} Lu = f & \textit{in } U, \\ u = 0 & \textit{on } \partial U, \end{cases} \tag{5.31}$$

then $u \in H^2(U)$, and there exists a constant C depending on U and the coefficients of L such that

$$\|u\|_{H^2(U)} \leq C(\|u\|_{L^2(U)} + \|f\|_{L^2(U)}). \tag{5.32}$$

Proof. See Evans [1] Theorem 4 of §6.3.2. □

Remark. If $u \in H_0^1(U)$ is a unique weak solution of the BVP (5.31), by Theorem 5.14, we can simplify the estimate (5.32) to

$$\|u\|_{H^2(U)} \leq C\|f\|_{L^2(U)}.$$

Theorem 5.21 (Higher boundary regularity)**.** *Let $m \in \mathbb{N}_0$. Assume that $a^{ij}, b^i, c \in C^{m+1}(\overline{U})$ for all $i, j = 1, 2, \cdots, n$, and $f \in H^m(U)$. Assume further that ∂U is C^{m+2}. If $u \in H_0^1(U)$ is a weak solution of*

$$\begin{cases} Lu = f & \textit{in } U, \\ u = 0 & \textit{on } \partial U, \end{cases} \tag{5.33}$$

then $u \in H^{m+2}(U)$, and there exists a constant C depending on U, m and the coefficients of L such that

$$\|u\|_{H^{m+2}(U)} \leq C(\|u\|_{H^m(U)} + \|f\|_{L^2(U)}).$$

Proof. See Evans [1] Theorem 5 of §6.3.2. □

Theorem 5.22 (Infinite differentiability up to the boundary)**.** *Assume that $a^{ij}, b^i, c \in C^\infty(U) \cap L^\infty(\overline{U})$ for all $i, j = 1, 2, \cdots, n$ and $f \in C^\infty(\overline{U})$. If $u \in H_0^1(U)$ is a weak solution of the BVP*

$$\begin{cases} Lu = f & \textit{in } U, \\ u = 0 & \textit{on } \partial U, \end{cases} \tag{5.34}$$

then $u \in C^\infty(\overline{U})$.

Proof. By Theorem 5.21, $u \in H^m(U)$ for all integers $m \in \mathbb{N}$. According to Theorem 4.12, $u \in C^k(U)$ for each $k > \frac{n}{2}$ by modifying u on a Lebesgue null set if necessary, and hence $u \in C^\infty(U)$. □

5.6 Maximum Principles

In this section, we work with the elliptic operator of non-divergence form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u,$$

and derive the maximum principles. As before, we assume the uniform ellipticity and symmetry condition in Definition 5.7 holds. The maximum principles are based upon the observations that a C^2 function u attains a local maximum at a point x_0 in an open set U if and only if $Du(x_0) = 0$ and $D^2u(x_0) \preceq 0$.

Throughout this section, we require that our solutions u are at least C^2 so that it makes sense to consider the pointwise values of Du and D^2u. This assumption is satisfied under some regularity conditions on the coefficients of L and the domain U.

5.6.1 Weak Maximum Principles

The weak maximum principles identify the functions that attain their maximum on the boundary.

Theorem 5.23 (Weak maximum principle)**.** *Let U be a bounded open set, and let the zeroth-order coefficient of L be $c \equiv 0$ in U. Assume that $u \in C^2(U) \cap C(\overline{U})$ and $Lu \le 0$ in U. Then*

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

Proof. Step I. We first assume the strict inequality $Lu < 0$ in U. By uniform ellipticity of L, the matrix $A(x_0) = (a^{ij}(x_0))_{i,j=1}^{n}$ is positive definite, and we take the spectral decomposition

$$A(x_0) = \sum_{i=1}^{n} \lambda_i q_i q_i^\top,$$

where $q_1, \cdots, q_n \in \mathbb{R}^n$ form an orthonormal basis of $\mathbb{R}^n$, and the eigenvalues $\lambda_1, \cdots, \lambda_n \ge \theta > 0$. If there exists a point $x_0 \in U$ with $u(x_0) = \max_{\overline{U}} u$, we have $Du(x_0) = 0$, and $D^2u(x_0) \preceq 0$. Then

$$\begin{aligned} Lu(x_0) &= -\sum_{i,j=1}^{n} a_{ij}(x_0)u_{x_i x_j}(x_0) + \sum_{i=1}^{n} b^i(x)u_{x_i} \\ &= -\sum_{i,j=1}^{n} a_{ij}(x_0)u_{x_i x_j}(x_0) = -\mathrm{tr}\left(A(x_0)D^2u(x_0)\right) = -\sum_{i=1}^{n} \lambda_i q_i^\top D^2u(x_0)q_i \ge 0, \end{aligned}$$

contradicting our assumption $Lu < 0$ on U. Hence a strict subsolution u attains its maximum over $\overline{U}$ on ∂U.

Step II. For the general case, we let $\lambda > \|b\|_{L^\infty(U)}/\theta$, and define $u_\epsilon(x) = u(x) + \epsilon e^{\lambda x_1}$ for $x \in U$. Then

$$Lu_\epsilon(x) = Lu(x) - \epsilon\lambda^2 a^{11}(x)e^{\lambda x_1} + \epsilon\lambda b^1(x)e^{\lambda x_1} \le -\epsilon\lambda e^{\lambda x_1}(\lambda\theta - b^1(x)) < 0$$

for each $\epsilon > 0$ and our choice of λ. By Step I, we have $\max_{\overline{U}} u_\epsilon = \max_{\partial U} u_\epsilon$, which implies and

$$\max_{\overline{U}} u + \epsilon e^{-\lambda\,\mathrm{diam}(U)} \le \max_{\partial U} u + \epsilon e^{\lambda\,\mathrm{diam}(U)}.$$

Passing $\epsilon \downarrow 0$ implies that

$$\max_{\overline{U}} u \le \max_{\partial U} u.$$

Since $\partial U \subset \overline{U}$, we also have $\max_{\overline{U}} u \ge \max_{\partial U} u$, which concludes the proof. $\square$

Remark. (i) A function satisfying $Lu \le 0$ in U is called a *subsolution.* We are thus asserting that *a subsolution attains its maximum on ∂U.* Similarly, a function satisfying $Lu \ge 0$ in U is called a *supersolution.* If the supersolution $u \in C^2(U) \cap C(\overline{U})$, we may apply this result to $-u$ to obtain

$$\min_{\overline{U}} u = \min_{\partial U} u.$$

(ii) If $Lu = 0$ in U, we have

$$\max_{\overline{U}} |u| = \max_{\partial U} |u|.$$

(iii) According to our proof, a strict subsolution $Lu < 0$ in U has no local minimum in U.

We can modify the weak maximum principle to allow for a nonnegative zeroth-order coefficient.

Theorem 5.24 (Weak maximum principle)**.** *Let U be a bounded open set, and let the zeroth-order coefficient of L be $c \ge 0$ in U. Assume that $u \in C^2(U) \cap C(\overline{U})$ and $Lu \le 0$ in U. Then*

$$\max_{\overline{U}} u \le \max_{\partial U} u^+. \tag{5.35}$$

Furthermore, the equality holds if $\sup_U u > 0$.

Proof. Let u be a subsolution, and define $Ku = Lu - cu$ on the bounded open set $V = \{x \in U : u(x) > 0\}$. Then K has no zeroth-order term, and $Ku \le 0$ on V. By Theorem 5.23, $\max_{\overline{V}} u = \max_{\partial V} u$.

If V is empty, we have $u \le 0$ on U, and the inequality (5.35) is trivial. If V is nonempty, we have $\max_{\partial V} u > 0$. We claim that $\{x \in \partial U : u(x) > 0\} = \{x \in \partial V : u(x) > 0\}$:

- If $x \in \partial U$ and $u(x) > 0$, by continuity of u, there exists $\epsilon > 0$ such that $u > 0$ on $\overline{U} \cap B(x, \epsilon)$, and $U \cap B(x, \epsilon) \subset V$. Then $x \in \overline{V}$. Since $x \notin V$, it must be the case $x \in \partial V$.
- If $x \in \partial V$ and $u(x) > 0$, it is clear that $x \in \overline{U}$. If $x \in U$, there exists $\epsilon > 0$ such that $u > 0$ on $B(x, \epsilon) \subset U$, and $B(x, \epsilon) \subset V$, contradicting the fact $x \in \partial V$. Hence $x \in \partial U$.

Therefore $\max_{\overline{U}} u = \max_{\overline{V}} u = \max_{\partial V} u = \max_{\partial U} u$, which concludes the proof. □

Remark. Similarly, if $u \in C^2(U) \cap C(\overline{U})$ and $Lu \ge 0$ in U, we have

$$\min_{\overline{U}} u \ge -\max_{\partial U} u^-.$$

In particular, if $Lu = 0$ in U, we combining the last two results to obtain

$$\max_{\overline{U}} |u| = \max_{\partial U} |u|.$$

Following is an immediate consequence of the weak maximum principle.

Corollary 5.25 (Uniqueness of solutions to the Dirichlet problem)**.** *Let the zeroth-order coefficient of L be $c \ge 0$ in U. Let $g \in C(\partial U)$. The Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } U, \\ u = g & \text{on } \partial U \end{cases} \tag{5.36}$$

has at most one solution in $C^2(U) \cap C(\overline{U})$, i.e. there may be no solution or a unique solution but cannot be two or more solutions.

Proof. Let $u_1, u_2 \in C^2(U) \cap C(\overline{U})$ be two solutions to (5.36). Then $L(u_1 - u_2) = 0$ and $u_1 - u_2 \equiv 0$ on ∂U. By the weak maximum principle, $\max_{\overline{U}} |u| = \max_{\partial U} |u| = 0$ on $\overline{U}$. Hence $u \equiv 0$ on $\overline{U}$. □

5.6.2 Strong Maximum Principles

We next substantially strengthen the foregoing assertions, by demonstrating that a subsolution u cannot attain its maximum at an interior point of a connected region at all, unless u is constant. Before we proceed, we introduce a technical lemma.

Lemma 5.26 (Hopf's lemma)**.** *Let U be a bounded open set, and let the zeroth-order coefficient of L be $c \equiv 0$ in U. Assume that*

(i) $u \in C^2(U) \cap C(\overline{U})$;

(ii) $Lu \leq 0$ in U, and there exists a point $x^0 \in \partial U$ such that $u(x^0) > u(x)$ for all $x^0 \in U$; and

(iii) U satisfies the interior ball condition at x^0, that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$.

Then

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

where $\nu \in \mathbb{R}^n$ is the outer unit normal to B at x^0. Furthermore, if we relax our assumption by only requiring that the zeroth-order coefficient $c \geq 0$ in U, then the same conclusion holds provided that $u(x^0) \geq 0$.

Proof. Step I. Let $B(y, r)$ be an open ball such that $x^0 \in \partial B(y, r)$ and $B(y, r) \subset U$. Define

$$v(x) = e^{-\lambda|x-y|^2} - e^{-\lambda r^2}, \quad x \in B(y, r),$$

where $\lambda > 0$ is to be selected below. We also assume $c \geq 0$ in U. By uniform ellipticity,

$$\begin{aligned} Lv(x) &= -4\lambda^2 e^{-\lambda|x-y|^2}(x-y)^\top A(x)(x-y) + 2\lambda e^{-\lambda|x-y|^2} \operatorname{tr} A(x) - 2\lambda e^{-\lambda|x-y|^2} b(x)^\top (x-y) + c(x)v(x) \\ &\leq e^{-\lambda|x-y|^2}\left(-4\theta\lambda^2|x-y|^2 + 2\lambda \operatorname{tr} A(x) - 2\lambda|b(x)|\,|x-y| + c(x)\right). \end{aligned}$$

In the open annular region $D = B(x, r) \backslash \overline{B(x, \frac{r}{2})}$,

$$Lv(x) \leq e^{-\lambda|x-y|^2}\left(-\theta\lambda^2 r^2 + 2\lambda \operatorname{tr} A(x) - 2\lambda|b(x)|\, r + c(x)\right).$$

By choosing 0 large enough, we have $Lv < 0$ in D.

Step II. Since $u(x^0) > u(x)$, there exists $\epsilon > 0$ so small that $u(x^0) \geq u(x) + \epsilon v(x)$ for $x \in \partial B(x, \frac{r}{2})$. Also note that $u(x^0) \geq u(x) + \epsilon v(x)$, since $v \equiv 0$ on $\partial B(x, r)$. Then

$$u(x) + \epsilon v(x) - u(x^0) \leq 0 \quad \text{on } \partial D.$$

Meanwhile, by Step I, if $u(x^0) \geq 0$ or $c \equiv 0$ on U, we have

$$L(u + \epsilon v - u(x^0)) = Lu + \epsilon Lv - cu(x^0) \leq -cu(x^0) \leq 0 \quad \text{in } D.$$

We apply the weak maximum principle [Theorem 5.24] to obtain that $u + \epsilon v - u(x^0) \leq 0$ in D. Note that $u(x^0) + \epsilon v(x^0) - u(x^0) = 0$. Hence

$$\frac{\partial u}{\partial \nu}(x^0) + \epsilon \frac{\partial v}{\partial \nu}(x^0) \geq 0$$

Consequently,

$$\frac{\partial u}{\partial \nu}(x^0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x^0) = \epsilon\nu \cdot 2\lambda(x_0 - y)e^{-\lambda|x_0-y|^2} = 2\epsilon\lambda r e^{-\lambda r^2} > 0,$$

as desired. ∎

The Hopf's lemma is an important tool in the proof of the strong maximum principle.

Theorem 5.27 (Strong maximum principle)**.** *Let U be an open, connected set. Assume $u \in C^2(U) \cap C(\overline{U})$, $Lu \le 0$ in U, and there exists $x^* \in U$ such that*

$$u(x^*) = \max_{\overline{U}} u.$$

(i) If the zeroth-order coefficient $c \equiv 0$ in U, then u is a constant within U.

(ii) If the zeroth-order coefficient $c \ge 0$ in U, and $u(x^*) \ge 0$, then u is a constant within U.

Proof. We write $M = \max_{\overline{U}} u$. If $u \not\equiv M$ on U, we take $z \in E$ with $u(z) < M$, and let E be the connected component of $\{x \in U : u(x) < M\}$ that contains z. Then $E \subset U$ is an open set, and $E \ne U$ since $x^* \notin E$. We also note that $\partial E \backslash \partial U$ is nonempty, since both E and U is connected. We fix $x_1 \in \partial E \backslash \partial U$ and $\epsilon > 0$ with $B(x_1, \epsilon) \subset U$. Then $y \in B(x_1, \frac{\epsilon}{2}) \cap E$ satisfies $d(y, \partial E) < \frac{\epsilon}{2} < d(y, \partial U)$.

Next, we let $B(y, r)$ be the largest open ball lying in E, and take $x^0 \in \partial B(y, r) \cap \partial E$. By definition, $u(x^0) = M > u(x)$ for all $x \in B(y, r)$. We then apply Hopf's lemma to obtain that $\frac{\partial u}{\partial \nu}(x^0) = \nu \cdot Du(x^0) > 0$. On the other hand, since u attains its maximum at $x^0 \in U \backslash E$, and $Du(x^0) = 0$, which is a contradiction. □

Remark. (i) Let the zeroth-order coefficient of L be $c \equiv 0$ (resp. $c \ge 0$) in U. Assume that $u \in C^2(U) \cap C(\overline{U})$ and $Lu \ge 0$ in U. If there exists $x^* \in U$ such that

$$u(x^*) = \min_{\overline{U}} u$$

(add condition $u(x^*) \le 0$ for $c \ge 0$), then u is a constant within U.

(ii) In this theorem, we do not require U to be bounded. Therefore, the weak maximum principle for unbounded sets, for example, the half space $U = \{x \in \mathbb{R}^n : x_1 > 0\}$, where $\partial U = \{x \in \mathbb{R}^n : x_1 = 0\}$.

5.6.3 Harnack's inequality

Harnack's inequality asserts that the values of a nonnegative solution of a linear elliptic PDE are comparable, at least in any subregion away from the boundary. For simplicity, we work with elliptic operators of the form

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j},$$

where the coefficients a^{ij} $(i, j = 1, \cdots, n)$ are smooth.

Theorem 5.28 (Harnack's inequality)**.** *Let $V \Subset U$ be connected. Then for each $u \in C^2(U)$ with $u \ge 0$ in U and $Lu = 0$ in U,*

$$\sup_V u \le C \inf_V u,$$

where C is a constant depending only on V and the coefficients of L.

Proof. We may assume $u > 0$ in U, for otherwise we could apply the result to $u + \epsilon$ and then let $\epsilon \downarrow 0$. □

5.7 Eigenvalues and Eigenfunctions

In this section, we consider the eigenvalue problem

$$\begin{cases} Lw = \lambda w & \text{in } U, \\ w = 0 & \text{on } \partial U, \end{cases} \tag{5.37}$$

where U is an open bounded region, and λ is an eigenvalue of differential operator L provided there is a nontrivial solution of (5.37). For simplicity, we work with symmetric elliptic operators of the form

$$Lu = \sum_{i,j=1}^{n} \left(a^{ij}(x) u_{x_i}\right)_{x_j}. \tag{5.38}$$

The associated bilinear form is

$$B(u,v) = \int_U \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} v_{x_j}\, dx.$$

Theorem 5.29 (Eigenvalues of symmetric elliptic operators). *Let L be a uniformly elliptic operator of the form* (5.38)*, where $a_{ij} \in C^1(\overline{U})$ for all $i,j = 1,2,\cdots$, and ∂U is C^2.*

(i) All eigenvalues of L are real with finite multiplicity;

(ii) If we repeat each eigenvalue according to its finite multiplicity, we have $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$ with

$$0 < \lambda_1 \le \lambda_2 \le \cdots,$$

and $\lambda_k \uparrow \infty$ as $k \to \infty$;

(iii) Finally, there exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(U)$, where $w_k \in H_0^1(U)$ is an eigenfunction corresponding to λ_k:

$$\begin{cases} Lw_k = \lambda_k w_k & \text{in } U, \\ w_k = 0 & \text{on } \partial U, \end{cases} \quad k = 1,2,\cdots.$$

According to the regularity theory discussed before, $w_k \in H^2(U)$ for $k = 1,2,\cdots$.

Proof. Step I. Recalling the proof of Theorem 5.8, we have $\gamma = 0$ in the energy estimate for symmetric elliptic operator L. By Theorem 5.9, for every $f \in L^2(U)$, there is a unique $u \in H_0^1(U)$ solving $B(u,v) = \langle f, v\rangle_{L^2(U)}$ for all $v \in H_0^1(U)$. Furthermore, following the proof of Theorem 5.12, the inverse K defined by $Kf = u$ is a compact bounded linear operator mapping $L^2(U)$ into itself.

Step II. We claim that K is self-adjoint. Let $f,g \in L^2(U)$. Then $u = Kf$ is a weak solution of the corresponding elliptic BVP, and $B(u,v) = \langle f,v\rangle_{L^2(U)}$ for all $v \in H_0^1(U)$. We choose $v = Kg$, so $B(v,u) = \langle g,u\rangle_{L^2(U)}$. Hence

$$\langle f, Kg\rangle_{L^2(U)} = \langle f, v\rangle_{L^2(U)} = B(u,v) = B(v,u) = \langle g,u\rangle_{L^2(U)} = \langle g, Kf\rangle_{L^2(U)}.$$

Step III. According to the spectral theory for compact self-adjoint operators, K has at most countably many non-zero eigenvalues, each with a finite-dimensional eigenspace. Also note that

$$\langle f, Kf\rangle_{L^2(U)} = B(u,u) \ge 0,$$

and 0 is not an eigenvalue of K. Hence all eigenvalues of K are positive. We write $\eta_1 \ge \eta_2 \ge \cdots > 0$ for the eigenvalues of K, and write w_k for the corresponding normalized eigenfunctions, which form an orthonormal basis of $L^2(U)$. Since $L^2(U)$ is infinite-dimensional, L has infinitely many eigenvalues, and $\eta_k \downarrow 0$ as $k \to \infty$.

Step IV. Let $\eta \neq 0$. Then

$$Kw = \eta w \quad \Leftrightarrow \quad B(\eta w, v) = \langle w, v\rangle_{L^2(U)}, \ \forall v \in H_0^1(U) \quad \Leftrightarrow \quad Lw = \frac{w}{\eta}.$$

Hence the eigenvalues of L are $\lambda_k = 1/\eta_k$, $k = 1, 2, \cdots$, with $\lambda_k \uparrow \infty$ as $k \to \infty$, and $Lw_k = \lambda_k w_k$. $\square$

Remark. The eigenfunctions $\left(\frac{w_k}{\sqrt{\lambda_k}}\right)_{k=1}^{\infty}$ form an orthonormal basis of $H_0^1(U)$ under the inner product $B(\cdot,\cdot)$. Since $Lw_k = \lambda_k w_k$, we have

$$\begin{aligned} B(w_k, w_k) &= \langle Lw_k, w_k\rangle_{L^2(U)} = \lambda_k \|w_k\|_{L^2(U)}^2 = \lambda_k, \\ B(w_k, w_l) &= \langle Lw_k, w_l\rangle_{L^2(U)} = \lambda_k \langle w_k, w_l\rangle_{L^2(U)} = 0, \quad k \neq l. \end{aligned}$$

By classical Poincare's inequality and uniform ellipticiy, $H_0^1(U)$ is a Hilbert space under $B(\cdot,\cdot)$. If $u \in H_0^1(U)$ and $B(w_k, u) = 0$ for all $k \in \mathbb{N}$, we have

$$\langle w_k, u\rangle_{L^2(U)} = \frac{B(w_k, u)}{\lambda_k} = 0, \quad k = 1, 2, \cdots.$$

Noticing that $(w_k)_{k=1}^{\infty}$ is an orthonormal basis of $L^2(U)$, we have $u = 0$. Therefore $(w_k)_{k=1}^{\infty}$ is an orthogonal basis of $H_0^1(U)$ under $B(\cdot,\cdot)$, and

$$u = \sum_{k=1}^{\infty} B\left(u, \frac{w_k}{\sqrt{\lambda_k}}\right) \frac{w_k}{\sqrt{\lambda_k}},$$

where the series converges in $H_0^1(U)$.

In particular, the eigenfunctions $\left(\frac{w_k}{\sqrt{1+\lambda_k}}\right)_{k=1}^{\infty}$ of $L = -\Delta$ form an orthonormal basis of $H_0^1(U)$. Using integration by parts, we have

$$\begin{aligned} &\int_U |\nabla w_k|^2 \, dx = -\int_U w_k \Delta w_k \, dx = \lambda \int_U w_k^2 \, dx = \lambda_k, \\ &\int_U \nabla w_k \cdot \nabla w_l \, dx = -\int_U w_k \Delta w_l \, dx = \lambda_l \int_U w_k w_l \, dx = 0, \quad k \neq l. \end{aligned}$$

Following the same procedure as above, we see that $(w_k)_{k=1}^{\infty}$ is an orthogonal basis of $H_0^1(U)$.

We call the smallest eigenvalue $\lambda_1 > 0$ the *principal eigenvalue* of L.

Theorem 5.30 (Variational principle for the principal eigenvalue)**.** *Let λ_1 be the principal eigenvalue of L.*

(i) (Rayleigh's formula)

$$\lambda_1 = \min\left\{B(u, u) : u \in H_0^1(U), \|u\|_{L^2(U)} = 1\right\}. \tag{5.39}$$

(ii) Furthermore, the above minimum is attained for a function w_1, positive within U, which solves

$$\begin{cases} Lw_1 = \lambda_1 w_1 & \text{in } U, \\ w_1 = 0 & \text{on } U. \end{cases}$$

(iii) Finally, if $u \in H_0^1(U)$ is any weak solution of

$$\begin{cases} Lu = \lambda_1 u & \text{in } U, \\ u = 0 & \text{on } U, \end{cases} \tag{5.40}$$

then u is a multiple of w_1.

Proof. Step I. We let $(w_k)_{k=1}^\infty$ be the normalized eigenfunctions of L, which form an orthonormal basis of $L^2(U)$ and an orthonormal basis of $H_0^1(U)$ under $B(\cdot,\cdot)$. If $u \in H_0^1(U)$ and $\|u\|_{L^2(U)} = 1$, we write

$$u = \sum_{k=1}^\infty \langle u, w_k\rangle_{L^2(U)} w_k = \sum_{k=1}^\infty B\left(u, \frac{w_k}{\sqrt{\lambda_k}}\right)\frac{w_k}{\sqrt{\lambda_k}}.$$

Since $\|u\|_{L^2(U)} = 1$, we have

$$1 = \sum_{k=1}^\infty \langle u, w_k\rangle_{L^2(U)}^2 = \sum_{k=1}^\infty \frac{\langle u, w_k\rangle_{L^2(U)}}{\sqrt{\lambda_k}} B\left(u, \frac{w_k}{\sqrt{\lambda_k}}\right) \le \sum_{k=1}^\infty \frac{1}{\lambda_1}\langle u, w_k\rangle_{L^2(U)} B\left(u, w_k\right) = \frac{B(u,u)}{\lambda_1}.$$

Therefore $B(u,u) \ge \lambda_1$, and the inequality holds if $u = w_1$. This proves (i).

Step II. Let $u \in H_0^1(U)$ and $\|u\|_{L^2(U)} = 1$. We claim that u is a weak solution of (5.40) if and only if $B(u,u) = \lambda_1$. Clearly, (5.40) implies $B(u,u) = \langle \lambda_1 u, u\rangle_{L^2(U)} = \lambda_1\|u\|_{L^2(U)}^2 = \lambda_1$. Conversely, if $B(u,u) = \lambda_1$,

$$\sum_{k=1}^\infty \lambda_1\langle u, w_k\rangle_{L^2(U)}^2 = \lambda_1 = B(u,u) = \sum_{k=1}^\infty \langle u, w_k\rangle_{L^2(U)} B(u, w_k) = \sum_{k=1}^\infty \lambda_k\langle u, w_k\rangle_{L^2(U)}^2.$$

Consequently, $\langle u, w_k\rangle_{L^2(U)} = 0$ for all $\lambda_k > \lambda_1$. Since λ_1 has finite multiplicity, it follows that

$$u = \sum_{k=1}^m \langle u, w_k\rangle_{L^2(U)} w_k$$

for some m, and $Lu = \sum_{k=1}^m \langle u, w_k\rangle_{L^2(U)} Lw_k = \sum_{k=1}^m \langle u, w_k\rangle_{L^2(U)}\lambda_1 w_k = \lambda_1 u$, with $u = 0$ on ∂U.

Step III. We prove that if $u \in H_0^1(U)$ is a nontrivial weak solution of (5.40), then either $u > 0$ in U or $u < 0$ in U. We may assume $\|u\|_{L^2(U)} = 1$, and take

$$\alpha = \int_U (u^+)^2\,dx, \quad \beta = \int_U (u^-)^2\,dx.$$

Then $\alpha + \beta = 1$. Furthermore, by Proposition 2.8, $u^\pm \in H_0^1(U)$, with

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u \ge 0\}, \\ 0 & \text{a.e. on } \{u \le 0\}, \end{cases} \quad \text{and} \quad Du^- = \begin{cases} 0 & \text{a.e. on } \{u \ge 0\}, \\ -Du & \text{a.e. on } \{u \le 0\}. \end{cases}$$

Then $B(u^+, u^-) = B(u^-, u^+) = 0$, and

$$\lambda_1 = B(u,u) = B(u^+,u^+) + B(u^-,u^-) \ge \lambda_1\|u^+\|_{L^2(U)}^2 + \lambda_1\|u^-\|_{L^2(U)}^2 = \lambda_1(\alpha+\beta) = \lambda_1.$$

The above inequality is in fact an equality, and

$$B(u^+,u^+) = \lambda_1\|u^+\|_{L^2(U)}^2, \quad B(u^-,u^-) = \lambda_1\|u^-\|_{L^2(U)}^2$$

Hence both u^+ and u^- solves (5.40) in the weak sense. Since the coefficients of L are in $C^\infty(\overline{U})$, we have $u^+, u^- \in C^\infty(\overline{U})$. Note that $Lu^+ = \lambda_1 u^+ \ge 0$ on U, by the strong maximum principle, we have either $u > 0$ in U or $u \equiv 0$ in U. The same conclusion holds for u^-. This proves (ii).

Step IV. Let u and $\widetilde{u}$ be two nontrivial weak solution of (5.40). By Step III,

$$\int_U \widetilde{u}\,dx \neq 0.$$

Then there exists a constant c such that

$$\int_U (u - c\widetilde{u})\,dx = 0.$$

Note that $u - c\widetilde{u}$ is also a weak solution of (5.40). By Step III, $u \equiv c\widetilde{u}$ in U. This proves (iii). $\square$

Remark. The assertion (iii) says the eigenspace of λ_1 is one-dimensional, and $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$.

Theorem 5.31 (Courant's minimax principle)**.** *Let $0 < \lambda_1 \le \lambda_2 \le \lambda_3 < \cdots$ be the eigenvalues of elliptic operator L with zero Dirichlet boundary condition. Then*

$$\lambda_k = \min_{S \in \Sigma_{k-1}} \max_{\substack{u \in S^\perp \\ \|u\|_{L^2(U)}=1}} B(u,u), \quad k = 1, 2, \cdots,$$

where Σ_{k-1} is the collection of all $(k-1)$-dimensional subspaces of $H_0^1(U)$.

Proof. Let $S \in \Sigma_{k-1}$, and $W_k = \operatorname{span}\{w_1, \cdots, w_k\}$. Since S is $(k-1)$-dimensioanl, $S^\perp \cap W_k$ is a subspace with positive dimension. We take $u \in S^\perp \cap W_k$ with $\|u\|_{L^2(U)} = 1$. Then $u = \sum_{l=1}^k \langle u, w_l\rangle_{L^2(U)} w_l$, and

$$B(u,u) = \sum_{l=1}^k \langle u, w_l\rangle_{L^2(U)} B(u, w_l) = \sum_{l=1}^k \lambda_l \langle u, w_l\rangle_{L^2(U)}^2 \ge \lambda_k.$$

Hence for all $S \in \Sigma_{k-1}$,

$$\max_{\substack{u \in S^\perp \\ \|u\|_{L^2(U)}=1}} B(u,u) \ge \lambda_k. \tag{5.41}$$

On the other hand, if we take $S = \operatorname{span}\{w_1, \cdots, w_{k-1}\}$ and $u = w_k \in S^\perp$, we have

$$B(w_k, w_k) = \lambda_k \langle w_k, w_k\rangle_{L^2(U)}^2 = \lambda_k.$$

Therefore the lower bound in (5.41) can be reached, and

$$\lambda_k = \min_{S \in \Sigma_{k-1}} \max_{\substack{u \in S^\perp \\ \|u\|_{L^2(U)}=1}} B(u,u).$$

Thus we finish the proof. $\square$

6 Second-order Parabolic Equations

Setting. In this chapter, we study the second-order parabolic equations, which are natural generalizations of the heat equation. We assume U is an open and bounded set, and set $U_T = U \times (0, T]$ for some fixed time $T > 0$. We study the initial/boundary-value problem

$$\begin{cases} u_t + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0, T], \\ u = g & \text{on } U \times \{t = 0\}, \end{cases} \tag{6.1}$$

where $f : U_T \to \mathbb{R}$ and $g : U \to \mathbb{R}$ are given, and $u : \overline{U}_T \to \mathbb{R}$ is the unknown, written $u = u(x, t)$. The variable x taking value in $\overline{U}$ is called the *spatial variable*, and the variable t taking value in $[0, T]$ is called the *time variable*. Given coefficient functions $a^{ij}, b^i, c,\ (i, j = 1, \cdots, n)$, the second-order partial differential operator L is given by the either the divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u \tag{6.2}$$

or the non-divergence form

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u. \tag{6.3}$$

We give the definition of parabolic operators below.

Definition 6.1 (Uniformly parabolic operators)**.** Let L be a partial differential operator of either divergence form (6.2) or non-divergence form (6.3). Assume the coefficient functions $a^{ij}, b^i, c \in L^\infty(U_T)$ for all $i, j = 1, \cdots, n$, and also assume the symmetry condition

$$a^{ij} = a^{ji}, \quad i, j = 1, \cdots, n.$$

The differential operator $\frac{\partial}{\partial t} + L$ is said to be *(uniformly) parabolic*, if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$$

for all $(x, t) \in U_T$ and all $\xi \in \mathbb{R}^n$.

Remark. In particular, for each fixed time $0 \leq t \leq T$, the operator L is a uniformly elliptic operator in the spatial variable x.

General second-order parabolic equations describe in physical applications the time-evolution of the density of some quantity u, e.g. a chemical concentration, within the region U. The second-order term $\sum_{i,j=1}^{n} a^{ij}u_{x_i x_j}$ describes diffusion, the first-order term $\sum_{i=1}^{n} b^i u_{x_i}$, describes transport, and the zeroth-order term cu describes creation or depletion. A simplest example of second-order parabolic equation is the *heat equation*

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0, T], \\ u = g & \text{on } U \times \{t = 0\}, \end{cases} \tag{6.4}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator. In this example, $a_{ij} = \delta_{ij}, b_i = 0, c = 0$ for all $i, j = 1, \cdots, n$.

6.1 Banach Space-Valued Functions

In this section, we study a special kind of Sobolev spaces, which consist of functions mapping time into Banach spaces. For the completeness of our discussion, we first study the property of functions taking values in Banach spaces. We work with a Banach space X equipped with a norm $\|\cdot\|$. We will specify later the what are the elements of the space X.

6.1.1 Definition and Properties

Definition 6.2. Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and $(X, \|\cdot\|)$ a Banach space.

(i) A function $\boldsymbol{s} : \Omega \to X$ is said to be *simple* if it is of the form

$$\boldsymbol{s}(t) = \sum_{i=1}^{n} \chi_{E_i}(t) u_i,$$

where each E_i is a measurable subset of Ω with $m(E_i) < \infty$ and $u_i \in X$.

(ii) A function $\boldsymbol{u} : \Omega \to X$ is said to be *strongly measuable* if there exist simple functions $\boldsymbol{s}_k : \Omega \to X$ such that $\|\boldsymbol{u}(t) - \boldsymbol{s}_k(t)\| \to 0$ for a.e. $t \in \Omega$.

(iii) A function $\boldsymbol{u} : \Omega \to X$ is said to be *weakly measuable* if for every $f \in X^*$, the mapping $t \mapsto \langle f, \boldsymbol{u}(t)\rangle$ is a measurable function.

(iv) A function $\boldsymbol{u} : \Omega \to X$ is said to be *almost separably valued* if there exists a subset $E \subset \Omega$ with $m(E) = 0$ such that the set $\{\boldsymbol{u}(t) : t \in \Omega \backslash E\}$ is separable.

Remark. A strongly measurable function $\boldsymbol{u} : \Omega \to X$ must be weakly measurable. To see this, we take a sequence of simple functions $\boldsymbol{s}_k : \Omega \to X$ such that $\|\boldsymbol{u}(t) - \boldsymbol{s}_k(t)\| \to 0$. For each $f \in X^*$, the mapping $t \mapsto \langle f, \boldsymbol{s}_k(t)\rangle$ is of the form $\sum_{i=1}^{n} \chi_{E_i}(t)\langle f, u_i\rangle$, which is a simple function on Ω. Then the mapping $t \mapsto \langle f, \boldsymbol{u}(t)\rangle$ is a.e. the pointwise limit of a sequence of simple functions, which is measurable.

Also, a strongly measurable function $\boldsymbol{u} : \Omega \to X$ must be almost separably valued. To see this, we take S_k to be the range of $\boldsymbol{s}_k$, which is a finite set, and let E be the set of points $t \in \Omega$ such that $\boldsymbol{s}_k(t)$ does not converge to $\boldsymbol{u}(t)$. Then $m(E) = 0$, and $\{\boldsymbol{u}(t) : t \in \Omega \backslash E\} = \overline{\bigcup_{k=1}^{\infty} S_k}$.

We have the following criterion for strong measurability.

Theorem 6.3 (Pettis)**.** *A function $\boldsymbol{u} : \Omega \to X$ is strongly measurable if and only if it is weakly measurable and almost separably valued.*

Proof. We only need to show the "if" part. We may assume without loss of generality that $\{\boldsymbol{u}(t) : t \in \Omega\}$ is separable. We may also assume X is separable, else we can replace X by the closure of the range of $\boldsymbol{u}$.

Since X is separable, the closed unit ball in X^* is weak* separable. We take a sequence $(f_k) \subset X^*$ with $\|f_k\| \le 1$ such that for each $f \in X^*$ with $\|f\| \le 1$, there exists a subsequence (f_{k_j}) such that $\langle f_{k_j}, u\rangle \to \langle f, u\rangle$ for all $u \in X$. For any $\alpha \in \mathbb{R}$ and $f \in X^*$, we define

$$A = \{t \in \Omega : \|\boldsymbol{u}(t)\| \le \alpha\}, \quad \text{and} \quad A_f = \{t \in \Omega : |\langle f, \boldsymbol{u}(t)\rangle| \le \alpha\}.$$

It is clear that $A \subset \bigcap_{\|f\| \le 1} A_f$. On the other hand, by Hahn-Banach theorem, for each $t \in \Omega$, there exists $\|f_0\| = 1$ such that $\langle f_0, \boldsymbol{u}(t)\rangle = \|\boldsymbol{u}(t)\|$. Hence $A \supset \bigcap_{\|f\| \le 1} A_f$. Applying our density assertion, we have

$$A = \bigcap_{\|f\| \le 1} A_f = \bigcap_{k=1}^{\infty} A_{f_k}.$$

Since $\boldsymbol{u}$ is weakly measurable, every A_{f_k} is measurable, and the intersection A is also measurable. Hence the function $t \mapsto \|\boldsymbol{u}(t)\|$ is measurable.

Since the range of $\boldsymbol{u}$ is separable, for each $k \in \mathbb{N}$, we cover $\boldsymbol{u}(\Omega)$ by a sequence of open balls $B(u_{k,j}, \frac{1}{k})$. As before, the mapping $t \mapsto \|\boldsymbol{u}(t) - u_{k,j}\|$ is also measurable. Then the sets $B_{k,j} = \left\{t \in \Omega : \|\boldsymbol{u}(t) - u_{k,j}\| \leq \frac{1}{k}\right\}$ are measurable, with $\Omega = \bigcup_{j=1}^{\infty} B_{k,j}$. We set

$$\boldsymbol{u}_k(t) = u_{k,j}, \quad \text{if } t \in B'_{k,j} := B_{k,j} \backslash (B_{k,1} \cup \cdots \cup B_{k,j-1}).$$

By definition, we have $\|\boldsymbol{u}_k(t) - \boldsymbol{u}(t)\| \leq \frac{1}{k}$ for every $t \in \Omega$. Therefore $(\boldsymbol{u}_k)$ is a sequence of simple functions with strong limit $\boldsymbol{u}$, and $\boldsymbol{u}$ is strongly measurable. □

Next we define the integration of Banach space-valued functions.

Definition 6.4 (Bochner Integral)**.** For a simple function $\boldsymbol{s}(t) = \sum_{i=1}^{n} \chi_{E_i}(t) u_i$, define

$$\int_\Omega \boldsymbol{s}(t)\mu(dt) = \sum_{i=1}^{n} \mu(E_i) u_i.$$

A strongly measurable function $\boldsymbol{u} : \Omega \to X$ is said to be *Bochner integrable*, if there exists a sequence of simple functions $\boldsymbol{s}_k \to \boldsymbol{u}$ a.e. in such a way that

$$\lim_{n\to\infty} \int_\Omega \|\boldsymbol{u}(t) - \boldsymbol{s}_k(t)\| \, \mu(dt) = 0. \tag{6.5}$$

In that case, we define the *Bochner integral*

$$\int_\Omega \boldsymbol{u}(t)\mu(dt) = \lim_{n\to\infty} \int_\Omega \boldsymbol{s}_k(t)\mu(dt). \tag{6.6}$$

Remark. To justify this definition, we need to verify the limit on the right-hand side of (6.6) exists and is independent of the choice of approximation sequence $(\boldsymbol{s}_k)$. Note that

$$\left\| \int_\Omega \boldsymbol{s}_k(t)\mu(dt) - \int_\Omega \boldsymbol{s}_m(t)\mu(dt) \right\| \leq \int_\Omega \|\boldsymbol{s}_k(t) - \boldsymbol{s}_m(t)\|\mu(dt) \leq \int_\Omega \left(\|\boldsymbol{s}_k(t) - \boldsymbol{u}(t)\| + \|\boldsymbol{s}_m(t) - \boldsymbol{u}(t)\|\right) \mu(dt),$$

which converges to 0 as $k, m \to \infty$, and the limit exists by completeness of X. Also, the limit is independent of the choice of $(\boldsymbol{s}_k)$, since any two such sequences can be combined into a single approximating sequence.

Theorem 6.5 (Absolute integrability)**.** *A strongly measurable function $\boldsymbol{u} : \Omega \to X$ is Bochner integrable if and only if the function $\|\boldsymbol{u}\|$ is integrable. In that case,*

$$\left\| \int_\Omega \boldsymbol{u}(t)\mu(dt) \right\| \leq \int_\Omega \|\boldsymbol{u}(t)\|\mu(dt). \tag{6.7}$$

Proof. The "only if" part. Since $\boldsymbol{u}$ is strongly measurable, $t \mapsto \|\boldsymbol{u}(t)\|$ is measurable. By condition (6.5), we have $\int_\Omega \|\boldsymbol{s}_k(t) - \boldsymbol{u}(t)\| \, \mu(dt) < 1$ for large enough k, and

$$\int_\Omega \|\boldsymbol{u}(t)\|\mu(dt) \leq \int_\Omega \|\boldsymbol{s}_k(t)\|\mu(dt) + \int_\Omega \|\boldsymbol{s}_k(t) - \boldsymbol{u}(t)\|\mu(dt) < \infty.$$

The "if" part. Let $\boldsymbol{u}$ be a strongly measurable function such that $\|\boldsymbol{u}\|$ is integrable, and let $(\boldsymbol{u}_k)$ be a simple approximating sequence. Then $\boldsymbol{s}_k = \chi_{\{\|\boldsymbol{u}_k\| \leq 2\|\boldsymbol{u}\|\}} \boldsymbol{u}_k$ is also a simple approximating sequence such that $\boldsymbol{s}_k \leq 2\|u\|$. By dominated convergence theorem,

$$\lim_{k\to\infty} \int_\Omega \|\boldsymbol{u}(t) - \boldsymbol{s}_k(t)\|\mu(dt) = 0.$$

The final inequality is trivial for simple functions, and the general case follows by approximation. □

Corollary 6.6 (Dominated convergence theorem for the Bochner integral)**.** *Let $\boldsymbol{u}_k$ be a sequence of Bochner integrable functions such that $\boldsymbol{u}_k \to \boldsymbol{u}$ a.e.. If there exists an integrable function $g : \Omega \to \mathbb{R}_+$ such that $\|\boldsymbol{u}_k\| \le g$ a.e. for all k, then $\boldsymbol{u}$ is Bochner integrable and*

$$\int_\Omega \boldsymbol{u}(t)\mu(dt) = \lim_{k\to\infty} \int_\Omega \boldsymbol{u}_k(t)\mu(dt).$$

Proof. Since $\|\boldsymbol{u}\| \le g$ a.e., $\boldsymbol{u}$ is Bochner integrable. Note that $\|\boldsymbol{u} - \boldsymbol{u}_k\| \le 2g$. We then apply Theorem 6.5 and dominated convergence theorem to obtain

$$\lim_{k\to\infty} \left\| \int_\Omega \boldsymbol{u}(t)\mu(dt) - \int_\Omega \boldsymbol{u}_k(t)\mu(dt) \right\| \le \lim_{k\to\infty} \int_\Omega \|\boldsymbol{u}(t) - \boldsymbol{u}_k(t)\|\mu(dt) = 0.$$

Then we conclude the proof. □

Theorem 6.7. *Let X and Y be Banach spaces, and let $T : X \to Y$ be a bounded linear operator, and $\boldsymbol{u} : \Omega \to X$ a Bochner integrable function. Then $T\boldsymbol{u} : \Omega \to Y$ is Bochner integrable, and*

$$T \int_\Omega \boldsymbol{u}(t)\mu(dt) = \int_\Omega (T\boldsymbol{u})(t)\mu(dt). \tag{6.8}$$

Proof. Take simple functions $\boldsymbol{s}_k \to \boldsymbol{u}$ a.e.. Then $T\boldsymbol{s}_k$ is a simple approximating sequence of $T\boldsymbol{u}$, and $T\boldsymbol{u}$ is strongly measurable. Also note (6.8) is valid for simple functions, and the general case follows by definition. □

Remark. In particular, if $f \in X^*$, we have

$$\left\langle f, \int_\Omega \boldsymbol{u}(t)\mu(dt) \right\rangle = \int_\Omega \langle f, \boldsymbol{u}(t)\rangle \mu(dt).$$

6.1.2 Spaces Involving Time

In this section, we consider the time interval $\Omega = [0, T]$ with the Lebesgue measure. Generally, X is a real Banach space comprising functions on some measure space.

Definition 6.8 (L^p and C spaces involving time)**.** Let $T > 0$, and $(X, \|\cdot\|)$ a Banach space.

(i) Let $1 \le p < \infty$. The space $L^p(0, T; X)$ consists of all strongly measurable functions $\boldsymbol{u} : [0, T] \to X$ with

$$\|\boldsymbol{u}\|_{L^p(0,T;X)} := \left(\int_0^T \|\boldsymbol{u}(t)\|^p \, dt \right)^{1/p} < \infty.$$

The space $L^\infty(0, T; X)$ consists of all strongly measurable functions $\boldsymbol{u} : [0, T] \to X$ with

$$\|\boldsymbol{u}\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0\le t\le T} \|\boldsymbol{u}(t)\| < \infty.$$

(ii) The space $C([0, T]; X)$ consists of all continuous functions $\boldsymbol{u} : [0, T] \to X$ with

$$\|\boldsymbol{u}\|_{C([0,T];X)} := \sup_{0\le t\le T} \|\boldsymbol{u}(t)\| < \infty.$$

Definition 6.9 (Weak derivative)**.** Let $\boldsymbol{u} \in L^1(0, T; X)$. We say a function $\boldsymbol{v} : [0, T] \to X$ is the *weak derivative of* $\boldsymbol{u}$, written $\boldsymbol{u}' = \boldsymbol{v}$, if for all scalar test functions $\phi \in C_c^\infty([0, T])$,

$$\int_0^T \phi'(t)\boldsymbol{u}(t)\, dt = -\int_0^T \phi(t)\boldsymbol{v}(t)\, dt$$

Definition 6.10 (Sobolev spaces involving time)**.** Let $1 \le p \le \infty$. The Sobolev space $W^{1,p}(0,T;X)$ consists of all functions $\boldsymbol{u} \in L^p(0,T;X)$ such that $\boldsymbol{u}'$ exists in weak sense and belong to $L^p(0,T;X)$. We define

$$\|\boldsymbol{u}\|_{W^{1,p}(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T (\|\boldsymbol{u}(t)\|^p + \|\boldsymbol{u}'(t)\|^p)\, dt \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{0 \le t \le T} (\|\boldsymbol{u}(t)\| + \|\boldsymbol{u}'(t)\|), & p = \infty. \end{cases}$$

We also write $H^1(0,T;X) = W^{1,2}(0,T;X)$.

The space $W^{1,p}(0,T;X)$ can be continuously embedded into the space $C([0,T];X)$.

Proposition 6.11 (Calculus in an abstract space)**.** *Let $1 \le p \le \infty$, and $\boldsymbol{u} \in W^{1,p}(0,T;X)$. Then*

(i) $\boldsymbol{u} \in C([0,T];X)$ after possibly being redefined on a subset of $[0,T]$ of measure zero.

(ii) For all $0 \le s \le t \le T$,

$$\boldsymbol{u}(t) = \boldsymbol{u}(s) + \int_s^t \boldsymbol{u}'(\tau)\, d\tau.$$

(iii) There exists a constant C depending only on T such that

$$\sup_{0 \le t \le T} \|\boldsymbol{u}(t)\| \le C \|\boldsymbol{u}\|_{W^{1,p}(0,T;X)}.$$

Proof. We consider the mollification $\boldsymbol{u}^\epsilon = \eta_\epsilon * \boldsymbol{u}$, where η_ϵ is a mollifier on $\mathbb{R}$. Analogous to the Remark under Lemma 2.5, we can check that $(\boldsymbol{u}^\epsilon)' = \eta_\epsilon * \boldsymbol{u}'$. By Proposition 1.8 and the appending Remark, as $\epsilon \downarrow 0$, we have $\boldsymbol{u}^\epsilon \to \boldsymbol{u}$ a.e. on $[0,T]$, and $(\boldsymbol{u}^\epsilon)' \to \boldsymbol{u}'$ in $L^1(0,T;X)$. Given $0 < s < t < T$, we have

$$\boldsymbol{u}^\epsilon(t) = \boldsymbol{u}^\epsilon(s) + \int_s^t (\boldsymbol{u}^\epsilon)'(\tau)\, d\tau.$$

Letting $\epsilon \downarrow 0$, we have for a.e. $0 < s < t < T$ that

$$\boldsymbol{u}(t) = \boldsymbol{u}(s) + \int_s^t \boldsymbol{u}'(\tau)\, d\tau.$$

Since $\boldsymbol{u}' \in L^p(0,T;X) \subset L^1(0,T;X)$, the integral is continuous in both s and t. Hence $\boldsymbol{u}$ is in fact continuous on $[0,T]$, which gives both (i) and (ii). For the estimate (iii), the case $p = \infty$ is clear. If $1 \le p < \infty$, we write

$$\|\boldsymbol{u}(t)\| \le \left\| \boldsymbol{u}(s) + \int_s^t \boldsymbol{u}'(\tau)\, d\tau \right\| \le \|\boldsymbol{u}(s)\| + \int_s^t \|\boldsymbol{u}'(\tau)\|\, d\tau.$$

We integrate this relation with respect to s to obtain

$$\begin{aligned} T\|\boldsymbol{u}(t)\| &\le \int_0^T \|\boldsymbol{u}(s)\|\, ds + \int_0^T \int_s^t \|\boldsymbol{u}'(\tau)\|\, d\tau\, ds \\ &\le \int_0^T \|\boldsymbol{u}(s)\|\, ds + T \int_0^T \|\boldsymbol{u}'(\tau)\|\, d\tau \\ &\le T^{1-\frac{1}{p}} \|\boldsymbol{u}\|_{L^p(0,T;X)} + T^{2-\frac{1}{p}} \|\boldsymbol{u}'\|_{L^p(0,T;X)}. \end{aligned}$$

Since this estimate holds for all $t \in [0,T]$, the proof is completed. □

In the study of second order parabolic PDEs, we often work with the functions $\boldsymbol{u} \in L^2(0,T;H_0^1(U))$ for which $\boldsymbol{u}' \in L^2(0,T;H^{-1}(U))$. We have the following more specific results for these functions.

Theorem 6.12 (More calculus)**.** *Suppose $\boldsymbol{u} \in L^2(0,T;H_0^1(U))$ and $\boldsymbol{u}' \in L^2(0,T;H^{-1}(U))$. Then*

(i) $\boldsymbol{u} \in C([0,T];L^2(U))$ after possibly being redefined on a subset of $[0,T]$ of measure zero.

(ii) The mapping $t \mapsto \|\boldsymbol{u}(t)\|_{L^2(U)}$ is absolutely continuous, and

$$\frac{d}{dt}\|\boldsymbol{u}(t)\|_{L^2(U)} = 2\langle \boldsymbol{u}'(t), \boldsymbol{u}(t)\rangle$$

for a.e. $0 \le t \le T$, where $\langle \cdot,\cdot\rangle$ is the pairing between $H_0^1(U)$ and $H^{-1}(U)$.

(iii) There exists a constant C depending only on T such that

$$\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|_{L^2(U)} \le C\left(\|\boldsymbol{u}\|_{L^2(0,T;H_0^1(U))} + \|\boldsymbol{u}'\|_{L^2(0,T;H^{-1}(U))}\right).$$

Proof. We take the mollification $\boldsymbol{u}^\epsilon = \eta_\epsilon * \boldsymbol{u}$, where η_ϵ is a mollifier on $\mathbb{R}$. By the Remark under Lemma 2.5, Proposition 1.8 and the appending Remark, as $\epsilon \downarrow 0$, we have $\boldsymbol{u}^\epsilon \to \boldsymbol{u}$ a.e. on $[0,T]$ and in $L^2(0,T;H_0^1(U))$, and $(\boldsymbol{u}^\epsilon)' = \eta_\epsilon * \boldsymbol{u}' \to \boldsymbol{u}'$ in $L^2(0,T;H^{-1}(U))$. For any $0 \le t \le T$, we have

$$\begin{aligned}\frac{d}{dt}\left\|\boldsymbol{u}^\epsilon(t) - \boldsymbol{u}^\delta(t)\right\|_{L^2(U)} &= \frac{d}{dt}\int_U \left(\boldsymbol{u}^\epsilon(t) - \boldsymbol{u}^\delta(t)\right)^2 dx = \int_U 2\left[(\boldsymbol{u}^\epsilon)'(t) - (\boldsymbol{u}^\delta)'(t)\right]\cdot\left[\boldsymbol{u}^\epsilon(t) - \boldsymbol{u}^\delta(t)\right] dx\\ &= 2\left\langle (\boldsymbol{u}^\epsilon)'(t) - (\boldsymbol{u}^\delta)'(t), \boldsymbol{u}^\epsilon(t) - \boldsymbol{u}^\delta(t)\right\rangle,\end{aligned}$$

where we apply the dominated convergence theorem to interchange the differentiation and integration. Next, we fix $s \in [0,T]$ such that $\boldsymbol{u}^\epsilon(s) \to \boldsymbol{u}(s)$ in $L^2(U)$. Then

$$\begin{aligned}\left\|\boldsymbol{u}^\epsilon(t) - \boldsymbol{u}^\delta(t)\right\|_{L^2(U)} &\le \left\|\boldsymbol{u}^\epsilon(s) - \boldsymbol{u}^\delta(s)\right\|_{L^2(U)} + 2\int_0^T \left|\left\langle (\boldsymbol{u}^\epsilon)'(\tau) - (\boldsymbol{u}^\delta)'(\tau), \boldsymbol{u}^\epsilon(\tau) - \boldsymbol{u}^\delta(\tau)\right\rangle\right| d\tau\\ &\le \left\|\boldsymbol{u}^\epsilon(s) - \boldsymbol{u}^\delta(s)\right\|_{L^2(U)} + 2\int_0^T \left\|(\boldsymbol{u}^\epsilon)'(\tau) - (\boldsymbol{u}^\delta)'(\tau)\right\|_{H^{-1}(U)}\left\|\boldsymbol{u}^\epsilon(\tau) - \boldsymbol{u}^\delta(\tau)\right\|_{H_0^1(U)} d\tau\\ &\le \left\|\boldsymbol{u}^\epsilon(s) - \boldsymbol{u}^\delta(s)\right\|_{L^2(U)} + \left\|(\boldsymbol{u}^\epsilon)' - (\boldsymbol{u}^\delta)'\right\|_{L^2(0,T;H^{-1}(U))} + \left\|\boldsymbol{u}^\epsilon - \boldsymbol{u}^\delta\right\|_{L^2(0,T;H_0^1(U))},\end{aligned}$$

which holds for all $0 \le t \le T$. Therefore the mollifcation $(\boldsymbol{u}^\epsilon)$ is a Cauchy net in $C([0,T];L^2(U))$, which converges to some $\boldsymbol{v} \in C([0,T];L^2(U))$. Note that for a.e. $t \in [0,T]$, we have $\boldsymbol{u}^\epsilon(t) \to \boldsymbol{u}(t)$ in $H_0^1(U)$, and also in $L^2(U)$. Then we conclude that $\boldsymbol{u} = \boldsymbol{v}$ a.e., which gives (i). To show (ii), note that

$$\|\boldsymbol{u}^\epsilon(t)\|^2_{L^2(U)} = \|\boldsymbol{u}^\epsilon(s)\|^2_{L^2(U)} + 2\int_s^t |\langle (\boldsymbol{u}^\epsilon)'(\tau), \boldsymbol{u}^\epsilon(\tau)\rangle|\, d\tau.$$

Identifying $\boldsymbol{u}$ with $\boldsymbol{v}$ above and letting $\epsilon \downarrow 0$, we have for all $0 \le s \le t \le T$ that

$$\|\boldsymbol{u}(t)\|^2_{L^2(U)} \le \|\boldsymbol{u}(s)\|^2_{L^2(U)} + 2\int_s^t |\langle \boldsymbol{u}'(\tau), \boldsymbol{u}(\tau)\rangle|\, d\tau.$$

Finally, to show (iii), we integrate the above relation with respect to s to get

$$\begin{aligned}T\|\boldsymbol{u}(t)\|^2_{L^2(U)} &\le \int_0^T \|\boldsymbol{u}(s)\|^2_{L^2(U)}\, ds + 2\int_0^T\int_s^t |\langle \boldsymbol{u}'(\tau), \boldsymbol{u}(\tau)\rangle|\, d\tau\, ds\\ &\le \int_0^T \|\boldsymbol{u}(s)\|^2_{L^2(U)}\, ds + 2T\int_0^T \|\boldsymbol{u}'(\tau)\|_{H^{-1}(U)}\|\boldsymbol{u}(\tau)\|_{H_0^1(U)}\, d\tau\, ds\\ &\le \|\boldsymbol{u}\|^2_{L^2(0,T;L^2(U))} + T\left(\|\boldsymbol{u}\|^2_{L^2(0,T;H_0^1(U))} + \|\boldsymbol{u}'\|^2_{L^2(0,T;H^{-1}(U))}\right).\end{aligned}$$

Since this estimate holds for all $t \in [0,T]$, and $\|\boldsymbol{u}\|_{L^2(0,T;L^2(U))} \le \|\boldsymbol{u}\|_{L^2(0,T;H_0^1(U))}$, we conclude the proof. □

Theorem 6.13 (Mapping into better spaces)**.** *Let U be a bounded open set such that ∂U is smooth, and $m \in \mathbb{N}_0$. Suppose $\boldsymbol{u} \in L^2(0,T;H^{m+2}(U))$ and $\boldsymbol{u}' \in L^2(0,T;H^m(U))$. Then*

(i) $\boldsymbol{u} \in C([0,T];H^{m+1}(U))$ after possibly being redefined on a subset of $[0,T]$ of measure zero.

(ii) There exists a constant C depending only on T, U and m such that

$$\sup_{0\le t\le T} \|\boldsymbol{u}(t)\|_{H^{m+1}(U)} \le C\left(\|\boldsymbol{u}\|_{L^2(0,T;H^{m+2}(U))} + \|\boldsymbol{u}'\|_{L^2(0,T;H^m(U))}\right).$$

Proof. Step I. We first assume that $m=0$. We take a bounded open set $V \Supset U$, and apply Theorem 3.1 to construct an extension $E\boldsymbol{u} = \overline{\boldsymbol{u}}$ on $\mathbb{R}^n$, which compactly supported on V. In view of the estimate (3.4), we have $\overline{\boldsymbol{u}} \in L^2(0,T;H^2(V))$, and

$$\|\overline{\boldsymbol{u}}\|_{L^2(0,T;H^2(V))} \le C_1 \|\boldsymbol{u}\|_{L^2(0,T;H^2(U))}. \tag{6.9}$$

In addition, since E is a bounded linear operator from $L^2(U)$ into $L^2(V)$, we consider the difference quotients in variable t and apply methods similar to Theorem 5.16. We fix $\epsilon > 0$. Then for all $0 < |h| < \frac{\epsilon}{2}$,

$$\|D^h \boldsymbol{u}\|_{L^2(\epsilon,T-\epsilon;L^2(U))} \le \|\boldsymbol{u}'\|_{L^2(0,T;L^2(U))},$$

and

$$\|D^h \boldsymbol{u}\|_{L^2(\epsilon,T-\epsilon;L^2(V))} \le \|E\|_{L^2}\, \|D^h \boldsymbol{u}\|_{L^2(\epsilon,T-\epsilon;L^2(U))}.$$

We apply Theorem 5.16 (ii) and let $\epsilon \downarrow 0$ to get

$$\|\overline{\boldsymbol{u}}'\|_{L^2(0,T;L^2(V))} \le C_2 \|\boldsymbol{u}'\|_{L^2(0,T;L^2(U))}. \tag{6.10}$$

Step II. If $\overline{\boldsymbol{u}}$ is smooth, we apply integration by parts to obtain

$$\left|\frac{d}{dt}\int_V |D\overline{\boldsymbol{u}}(t)|^2\,dx\right| = 2\left|\int_V D\overline{\boldsymbol{u}}'(t)\cdot D\overline{\boldsymbol{u}}(t)\,dx\right| = 2\left|\int_V \overline{\boldsymbol{u}}'(t)\Delta\overline{\boldsymbol{u}}(t)\,dx\right| \le C_3\left(\|\overline{\boldsymbol{u}}'(t)\|_{L^2(V)} + \|\overline{\boldsymbol{u}}(t)\|_{H^2(V)}\right).$$

We then integrate on both sides with respect to t to get

$$\|D\overline{\boldsymbol{u}}(t)\|_{L^2(V)} \le C_4\left(\|\overline{\boldsymbol{u}}'\|_{L^2(0,T;L^2(V))} + \|\overline{\boldsymbol{u}}\|_{L^2(0,T;H^2(V))}\right).$$

Similarly,

$$\|\overline{\boldsymbol{u}}(t)\|_{L^2(V)} \le C_5\left(\|\overline{\boldsymbol{u}}'\|_{L^2(0,T;L^2(V))} + \|\overline{\boldsymbol{u}}\|_{L^2(0,T;L^2(V))}\right).$$

Recalling the estimates (6.9) and (6.10), we have

$$\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|_{H^1(U)} \le C_6\left(\|\boldsymbol{u}'\|_{L^2(0,T;L^2(U))} + \|\boldsymbol{u}\|_{L^2(0,T;H^2(U))}\right).$$

The same estimate holds even if $\boldsymbol{u}$ is not smooth, by approximating $\eta_\epsilon * \boldsymbol{u}$, as before. As in the previous proofs, it also follows that $\boldsymbol{u} \in C([0,T];H^1(U))$.

Step III. For the general case $m \ge 1$, we finish the proof by induction. Let α be a multiindex of order $|\alpha| \le m$, and set $\boldsymbol{v} = D^\alpha \boldsymbol{u}$. Then $\boldsymbol{v} \in L^2(0,T;H^2(U))$ and $\boldsymbol{v}' \in L^2(0,T;L^2(U))$. Then $\boldsymbol{v} \in C([0,T],H^1(U))$, and

$$\sup_{0\le t\le T}\|\boldsymbol{v}(t)\|_{H^1(U)} \le C\left(\|\boldsymbol{v}'\|_{L^2(0,T;L^2(U))} + \|\boldsymbol{v}\|_{L^2(0,T;H^2(U))}\right).$$

We take summation over all multi-indices $|\alpha| \le m$ to conclude the proof. $\square$

6.2 Weak Formulation of Second-order Parabolic Equations

In this section, we study the initial/boundary-value problem

$$\begin{cases} u_t + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0,T], \\ u = g & \text{on } U \times \{t=0\}, \end{cases} \tag{6.11}$$

where L is a uniformly parabolic operator of the divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x,t)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x,t)u_{x_i} + c(x,t)u. \tag{6.12}$$

To find an appropriate weak formulation for the initial/boundary-value problem 6.11, we assume for now that

$$a^{ij}, b^i, c \in L^\infty(U_T), \quad f \in L^2(U_T), \quad \text{and} \quad g \in L^2(U).$$

Definition 6.14. The time-dependent bilinear form $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ associated with the divergence form operator L defined by (5.2) is given by

$$B(u,v;t) = \int_U \left(\sum_{i,j=1}^{n} a^{ij}(\cdot,t)u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i(\cdot,t)u_{x_i}v + c(\cdot,t)uv \right) dx$$

for $u, v \in H_0^1(U)$ and a.e. $t \in [0,T]$.

Motivation. We assume that u is a smooth solution of the PDE (6.11). We switch our viewpoint by associate u with a mapping $\boldsymbol{u} : [0,T] \to H_0^1(U)$ defined by

$$[\boldsymbol{u}(t)](x) = u(x,t), \quad x \in U, \ 0 \le t \le T.$$

Also, we define $\boldsymbol{f} : [0,T] \to L^2(U)$ by

$$[\boldsymbol{f}(t)](x) = f(x,t), \quad x \in U, \ 0 \le t \le T.$$

If $v \in H_0^1(U)$, we multiply the PDE $u_t + Lu = f$ by v and integrate by parts to obtain

$$\langle \boldsymbol{u}'(t), v\rangle_{L^2(U)} + B(\boldsymbol{u}, v; t) = \langle \boldsymbol{f}(t), v\rangle_{L^2(U)}, \quad 0 \le t \le T. \tag{6.13}$$

Meanwhile, recalling Theorem 5.1, we have

$$u_t = g^0 + \sum_{j=1}^{n} g^j_{x_j} := \left(f - \sum_{i=1}^{n} b^i u_{x_i} - cu \right) - \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a^{ij} u_{x_i} \right)_{x_j} \in H^{-1}(U),$$

with the estimate

$$\|u_t\|_{H^{-1}(U)} \le \left(\int_U \sum_{j=0}^{n} |g^j|^2 \right)^{1/2} \le C\left(\|u\|_{H_0^1(U)} + \|f\|_{L^2(U)} \right).$$

This estimate suggests that it may be reasonable to find a weak solution with $\boldsymbol{u}' \in H^{-1}(U)$ for a.e. $0 < t \le T$, in which case the first term in (6.13) can be rewritten as $\langle \boldsymbol{u}'(t), v\rangle$, which is the pairing of $H^{-1}(U)$ and $H_0^1(U)$.

Definition 6.15 (Weak solutions)**.** Let L be a divergence form operator defined by (5.2), and let $B(\cdot,\cdot;t)$ be the associated time-dependent bilinear form. A function $\boldsymbol{u} \in L^2(0,T;H_0^1(U))$ with $\boldsymbol{u}' \in L^2(0,T;H^{-1}(U))$ is said to be a *weak solution* to the parabolic initial/boundary-value problem (6.11), if $\boldsymbol{u}(0) = g$, and

$$\langle \boldsymbol{u}'(t), v\rangle + B(\boldsymbol{u}, v; t) = \langle \boldsymbol{f}(t), v\rangle \tag{6.14}$$

for each $v \in H_0^1(U)$ and a.e. $0 \le t \le T$.

Remark. According to Theorem 6.12, we identify $\boldsymbol{u}$ with the continuous version $\boldsymbol{u} \in C([0,T];L^2(U))$, and thus the requirement $\boldsymbol{u}(0) = g$ makes sense.

Next, we study the existence and uniqueness of weak solutions of second-order parabolic PDEs.

6.2.1 Galerkin's Method

In this part, we build a weak solution of the parabolic initial/boundary-value problem (6.11) by constructing finite-dimensional approximations and passing to limits. This is called the *Galerkin's method.*

Approximation on finite basis. We take a collection of smooth functions $w_k = w_k(x)$ such that
(i) $(w_k)_{k=1}^\infty$ is an orthogonal basis of $H_0^1(U)$, and
(ii) $(w_k)_{k=1}^\infty$ is an orthonormal basis of $L^2(U)$.
For example, we can take $(w_k)_{k=1}^\infty$ to be the completed set of appropriately normalized eigenfunctions of the negative Laplacian operator $-\Delta$ in H_0^1.

We fix $m \in \mathbb{N}$, and seek a function $\boldsymbol{u}_m : [0,T] \to H_0^1(U)$ that can be seen as a projection of a solution of (6.11) onto the finite-dimensional subspace spanned by functions $(w_k)_{k=1}^m$. This projection is of the form

$$\boldsymbol{u}_m(t) = \sum_{k=1}^m d_m^k(t) w_k. \tag{6.15}$$

By definition of the weak solution, we select the coefficients d_m^k according to

$$\begin{cases} d_m^k(0) = \langle g, w_k\rangle_{L^2(U)}, \\ \langle \boldsymbol{u}_m'(t), w_k\rangle_{L^2(U)} + B(\boldsymbol{u}_m, w_k; t) = \langle \boldsymbol{f}(t), w_k\rangle_{L^2(U)}. \end{cases} \tag{6.16}$$

Theorem 6.16 (Construction of approximate solutions)**.** *For each $m \in \mathbb{N}$, there exists a function $\boldsymbol{u}_m$ of the form* (6.15) *that satisfies* (6.16)*.*

Proof. If $\boldsymbol{u}_m$ is of the form (6.15), by orthonormality of $(w_k)_{k=1}^\infty$,

$$(d_m^k)'(t) = \langle \boldsymbol{u}_m'(t), w_k\rangle_{L^2(U)}, \quad \text{and} \quad B(\boldsymbol{u}_m, w_k; t) = \sum_{l=1}^m e^{kl}(t) d_m^l(t),$$

where $e^{kl}(t) = B(w_l, w_k; t)$ for $k, l = 1, 2, \cdots$. We further write $f^k(t) = \langle \boldsymbol{f}(t), w_k\rangle_{L^2(U)}$ and $g^k = \langle g, w_k\rangle_{L^2(U)}$. Then (6.16) becomes a linear system of ODE

$$\begin{cases} (d_m^k)'(t) + \sum_{l=1}^m e^{kl}(t) d_m^l(t) = f^k(t), \\ d_m^k(0) = g^k, \end{cases} \qquad k = 1, 2, \cdots, m. \tag{6.17}$$

According to standard existence theory for ordinary differential equations, there exists a unique absolutely continuous function $\boldsymbol{d}_m(t) = (d_m^1(t), \cdots, d_m^m(t))$ satisfying (6.17) for a.e. $t \in [0,T]$. Hence the $\boldsymbol{u}_m$ defined by (6.15) solves (6.16) for a.e. $t \in [0,T]$. ☐

6.2.2 Energy Estimates

To establish an appropriate convergence result for the approximating sequence $(\boldsymbol{u}_m)_{m=1}^{\infty}$ constructed by Galerkin's method, we need some uniform estimates.

Theorem 6.17 (Energy estimates for the parabolic PDE)**.** *Let $(\boldsymbol{u}_m)_{m=1}^{\infty}$ be the approximating sequence* (6.15) *obtained by solving* (6.16)*. Then there exists a constant $C>0$, depending only on U, T and coefficients of L, such that for $m=1,2,\cdots$,*

$$\sup_{0\le t\le T}\|\boldsymbol{u}_m(t)\|_{L^2(U)}+\|\boldsymbol{u}_m\|_{L^2(0,T;H_0^1(U))}+\|\boldsymbol{u}_m'\|_{L^2(0,T;H^{-1}(U))}\le C\left(\|\boldsymbol{f}\|_{L^2(0,T;L^2(U))}+\|g\|_{L^2(U)}\right). \tag{6.18}$$

Proof. Step I. Combining (6.15) and (6.16), we have

$$\langle \boldsymbol{u}_m'(t),\boldsymbol{u}_m\rangle_{L^2(U)}+B(\boldsymbol{u}_m,\boldsymbol{u}_m;t)=\langle \boldsymbol{f}(t),\boldsymbol{u}_m\rangle_{L^2(U)} \tag{6.19}$$

for a.e. $t\in[0,T]$. By Theorem 5.8, there exist constants $\beta>0$ and $\gamma\ge 0$ such that

$$\beta\|\boldsymbol{u}_m(t)\|_{H_0^1(U)}^2\le B(\boldsymbol{u}_m,\boldsymbol{u}_m;t)+\gamma\|\boldsymbol{u}_m(t)\|_{L^2(U)}^2$$

for all $t\in[0,T]$ and $m\in\mathbb{N}$. Since $\frac{d}{dt}\|\boldsymbol{u}_m(t)\|_{L^2(U)}^2=2\langle \boldsymbol{u}_m'(t),\boldsymbol{u}_m\rangle_{L^2(U)}$, the estimate (6.19) implies

$$\frac{d}{dt}\|\boldsymbol{u}_m(t)\|_{L^2(U)}^2+2\beta\|\boldsymbol{u}_m(t)\|_{H_0^1(U)}^2\le C_1\left(\|\boldsymbol{u}_m(t)\|_{L^2(U)}^2+\|\boldsymbol{f}(t)\|_{L^2(U)}^2\right) \tag{6.20}$$

for all $t\in[0,T]$, where C_1 is a constant depends only on U and coefficients of L.

Step II. We relax estimate (6.20) to obtain

$$\|\boldsymbol{u}_m(t)\|_{L^2(U)}^2\le\|\boldsymbol{u}_m(0)\|_{L^2(U)}^2+\int_0^T\|\boldsymbol{f}(s)\|_{L^2(U)}^2\,ds+C_1\int_0^t\|\boldsymbol{u}_m(s)\|_{L^2(U)}^2\,ds.$$

By Grönwall's lemma, for all $0\le t\le T$,

$$\|\boldsymbol{u}_m(t)\|_{L^2(U)}^2\le e^{C_1t}\left(\|\boldsymbol{u}_m(0)\|_{L^2(U)}^2+\int_0^T\|\boldsymbol{f}(s)\|_{L^2(U)}^2\,ds\right).$$

Note that $\|\boldsymbol{u}_m(0)\|_{L^2(U)}\le\|g\|_{L^2(U)}$, we have

$$\sup_{0\le t\le T}\|\boldsymbol{u}_m(t)\|_{L^2(U)}^2\le e^{C_1T}\left(\|\boldsymbol{f}\|_{L^2(0,T;L^2(U))}^2+\|g\|_{L^2(U)}\right). \tag{6.21}$$

Step III. We integrate both sides of (6.20), and apply (6.21) to get

$$\|\boldsymbol{u}_m\|_{L^2(0,T;H_0^1(U))}^2=\int_0^T\|\boldsymbol{u}_m(t)\|_{H_0^1(U)}^2\,dt\le\frac{C_1(1+e^{C_1T})}{2\beta}\left(\|\boldsymbol{f}\|_{L^2(0,T;L^2(U))}^2+\|g\|_{L^2(U)}\right). \tag{6.22}$$

Step IV. We take any $v\in H_0^1(U)$, with $\|v\|_{H_0^1(U)}\le 1$, and write $v=v_0+v_1$, where $v_0=\sum_{k=1}^m\langle v,w_k\rangle_{L^2(U)}w_k$ is the projection of v onto $\operatorname{span}\{w_1,\cdots,w_m\}$, and $\langle v_1,w_k\rangle_{L^2(U)}=0$ for all $k=1,2,\cdots,m$. By (6.16),

$$\langle \boldsymbol{u}_m'(t),v_0\rangle_{L^2(U)}+B(\boldsymbol{u}_m,v_0;t)=\langle \boldsymbol{f}(t),v_0\rangle_{L^2(U)}$$

for a.e. $t\in[0,T]$. Since $\|v_0\|_{H_0^1(U)}\le\|v\|_{H_0^1(U)}\le 1$, we have

$$|\langle \boldsymbol{u}_m'(t),v\rangle|=\left|\langle \boldsymbol{u}_m'(t),v_0\rangle_{L^2(U)}\right|=\left|\langle \boldsymbol{f}(t),v_0\rangle_{L^2(U)}-B(\boldsymbol{u}_m,v_0;t)\right|\le C_2\left(\|\boldsymbol{f}(t)\|_{L^2(U)}+\|\boldsymbol{u}_m(t)\|_{H_0^1(U)}\right).$$

The above estimate holds for all $v \in H_0^1(U)$ with $\|v\|_{H_0^1(U)} \leq 1$. Hence

$$\|\boldsymbol{u}_m'(t)\|_{H^{-1}(U)} \leq C_2 \left(\|\boldsymbol{f}(t)\|_{L^2(U)} + \|\boldsymbol{u}_m(t)\|_{H_0^1(U)} \right).$$

We integrate this relation on $[0,T]$ and apply the estimate (6.22) to obtain

$$\|\boldsymbol{u}_m'\|_{L^2(0,T;H^{-1}(U))} \leq C_3 \left(\|\boldsymbol{f}\|_{L^2(0,T;L^2(U))}^2 + \|g\|_{L^2(U)} \right). \tag{6.23}$$

Combining (6.21), (6.22) and (6.23), we conclude the proof. □

6.2.3 Existence and Uniqueness

In this part, we pass m to infinity and show that a subsequence of the solutions $(\boldsymbol{u}_m)_{m=1}^\infty$ of the projected problem (6.16) converges to a weak solution of (6.11).

Theorem 6.18 (Existence theorem for weak solutions)**.** *There exists a weak solution of the parabolic initial/boundary-value problem* (6.11).

Proof. We take the approximating sequence $(\boldsymbol{u}_m)_{m=1}^\infty$ constructed by Galerkin's method.

Step I. By Theorem (6.18), the sequence $(\boldsymbol{u}_m)_{m=1}^\infty$ is bounded in $L^2(0,T;H_0^1(U))$, and the sequence $(\boldsymbol{u}_m')_{m=1}^\infty$ is bounded in $L^2(0,T;H^{-1}(U))$. By Banach-Alaoglu theorem, there exists a subsequence $(\boldsymbol{u}_{m_l})_{l=1}^\infty$ such that

- $(\boldsymbol{u}_{m_l})_{l=1}^\infty$ converges weakly to some function $\boldsymbol{u}$ in $L^2(0,T;H_0^1(U))$, and
- $(\boldsymbol{u}_{m_l}')_{l=1}^\infty$ converges weakly to some function $\boldsymbol{v}$ in $L^2(0,T;H^{-1}(U))$.

We claim that $\boldsymbol{u}' = \boldsymbol{v}$. Using integration by parts, for any $\phi \in C_c^\infty(0,T)$ and $h \in H_0^1(U)$,

$$\int_0^T \phi(t)\langle \boldsymbol{u}_{m_l}'(t), h\rangle \, dt = -\int_0^T \phi'(t)\langle \boldsymbol{u}_{m_l}(t), h\rangle \, dt.$$

Letting $l \to \infty$, the weak convergence implies

$$\int_0^T \phi(t)\langle \boldsymbol{v}(t), h\rangle \, dt = -\int_0^T \phi'(t)\langle \boldsymbol{u}(t), h\rangle \, dt,$$

which holds for all $h \in H_0^1(U)$. Hence

$$\int_0^T \phi(t)\boldsymbol{v}(t) \, dt = -\int_0^T \phi'(t)\boldsymbol{u}(t) \, dt$$

in $H^{-1}(U)$. Note that $\phi \in C_c^\infty(0,T)$, the result follows.

Step II. We fix an integer N, and take a function $\boldsymbol{v} \in C^1([0,T];H_0^1(U))$ of the form

$$\boldsymbol{v} = \sum_{k=1}^N d^k(t) w_k,, \tag{6.24}$$

where $(d^k)_{k=1}^N$ are given smooth functions. Recalling (6.16), for any $m \geq M$, we have

$$\int_0^T \left(\langle \boldsymbol{u}_m'(t), \boldsymbol{v}(t)\rangle_{L^2(U)} + B(\boldsymbol{u}_m, \boldsymbol{v}; t) \right) dt = \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{v}(t)\rangle_{L^2(U)} \, dt. \tag{6.25}$$

We let $m = m_l$ and pass $l \to \infty$. The weak convergence result implies

$$\int_0^T (\langle \boldsymbol{u}'(t), \boldsymbol{v}(t)\rangle + B(\boldsymbol{u}, \boldsymbol{v}; t)) \, dt = \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{v}(t)\rangle \, dt. \tag{6.26}$$

The above equality also holds for all $\boldsymbol{v} \in L^2(0,T;H_0^1(U))$ by applying the dominated convergence theorem, since the functions of the form (6.24) is dense in this space. In particular, for each $v \in H_0^1(U)$, plugging in $\boldsymbol{v}(t) = \phi(t)v$ to the last display for all $\phi \in C_c^\infty(0,T)$ gives

$$\langle \boldsymbol{u}'(t), v\rangle + B(\boldsymbol{u}, v; t) = \langle \boldsymbol{f}(t), v\rangle \quad \text{for a.e. } 0 \le t \le T.$$

By Theorem 6.12, we can identify $\boldsymbol{u} \in C([0,T];L^2(U))$.

Step III. It remains to verify that $\boldsymbol{u}(0) = g$. We take any $\boldsymbol{v} \in C^1([0,T];H_0^1(U))$ with $\boldsymbol{v}(T) = 0$. Applying integration by parts, (6.26) becomes

$$\int_0^T (-\langle \boldsymbol{v}'(t), \boldsymbol{u}(t)\rangle + B(\boldsymbol{u}, \boldsymbol{v}; t))\, dt = \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{v}(t)\rangle\, dt + \langle \boldsymbol{u}(0), \boldsymbol{v}(0)\rangle, \tag{6.27}$$

Also, (6.25) becomes

$$\int_0^T (-\langle \boldsymbol{v}'(t), \boldsymbol{u}_m(t)\rangle + B(\boldsymbol{u}_m, \boldsymbol{v}; t))\, dt = \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{v}(t)\rangle_{L^2(U)}\, dt + \langle \boldsymbol{u}_m(0), \boldsymbol{v}(0)\rangle.$$

Letting $m = m_l \to \infty$, we get

$$\int_0^T (-\langle \boldsymbol{v}'(t), \boldsymbol{u}(t)\rangle + B(\boldsymbol{u}, \boldsymbol{v}; t))\, dt = \int_0^T \langle \boldsymbol{f}(t), \boldsymbol{v}(t)\rangle_{L^2(U)}\, dt + \langle g, \boldsymbol{v}(0)\rangle. \tag{6.28}$$

Comparing (6.27) and (6.28), and noticing that $\boldsymbol{v}(0)$ is arbitrary, we conclude that $\boldsymbol{u}(0) = g$. $\square$

In addition, the weak solution of a parabolic PDE is unique.

Theorem 6.19 (Uniqueness of weak solutions)**.** *The weak solution of the parabolic PDE* (6.11) *is unique.*

Proof. It suffices to check that $\boldsymbol{u} \equiv 0$ is the only weak solution of the problem

$$\begin{cases} u_t + Lu = 0 & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0,T], \\ u = 0 & \text{on } U \times \{t = 0\}, \end{cases} \tag{6.29}$$

which is satisfied by the difference of any two weak solutions of (6.11). We set $v = \boldsymbol{u}(t)$ in (6.14). Then

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|_{L^2(U)} + B(\boldsymbol{u}, \boldsymbol{u}; t) = \langle \boldsymbol{u}'(t), \boldsymbol{u}(t)\rangle + B(\boldsymbol{u}, \boldsymbol{u}; t) = 0$$

for a.e. $0 \le t \le T$. Since

$$B(\boldsymbol{u}, \boldsymbol{u}; t) \ge \beta\|\boldsymbol{u}(t)\|_{H_0^1(U)} - \gamma\|\boldsymbol{u}\|_{L^2(U)}^2 \ge -\gamma\|\boldsymbol{u}\|_{L^2(U)}^2,$$

and $\boldsymbol{u}(0) = 0$, we have

$$\|\boldsymbol{u}(t)\|_{L^2(U)}^2 \le 2\gamma \int_0^t \|\boldsymbol{u}(s)\|_{L^2(U)}^2\, ds,$$

for all $0 \le t \le T$. By Gronwall's lemma, we immediately conclude $\boldsymbol{u} \equiv 0$. $\square$

6.3 Regularity Theory

In this section, we study the smoothness of the weak solutions of parabolic PDEs. We work with the parabolic operator of divergence form:

$$\frac{\partial}{\partial t} + L, \quad \text{where } Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu.$$

We further suppose that

- U is an open, bounded set with ∂U smooth, and
- the coefficients a^{ij}, b^i, c $(i, j = 1, \cdots, n)$ are smooth on $\overline{U}$ and not dependent on the time variable t.

For simplicity, we take $(\boldsymbol{u}_m)_{m=1}^{\infty}$ to be the eigenfunctions of $-\Delta$ on U, which form an orthonormal basis of $L^2(U)$ and an orthogonal basis of $H_0^1(U)$. Our analysis will be based on the weak solution constructed by Galerkin's method.

Theorem 6.20 (Improved regularity)**.** *Suppose that $\boldsymbol{u} \in L^2(0, T; H_0^1(U))$ with $\boldsymbol{u}' \in L^2(0, T; H^{-1}(U))$ is the weak solution of the initial/boundary-value problem*

$$\begin{cases} u_t + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0, T], \\ u = g & \text{on } U \times \{t = 0\}. \end{cases}$$

(i) Assume that $g \in H_0^1(U)$ and $\boldsymbol{f} \in L^2(0, T; L^2(U))$. Then

$$\boldsymbol{u} \in L^2(0, T; H^2(U)) \cap L^\infty(0, T; H_0^1(U)) \quad \text{and} \quad \boldsymbol{u}' \in L^2(0, T; L^2(U)).$$

Furthermore, we have the estimate

$$\operatorname*{ess\,sup}_{0 \le t \le T} \|\boldsymbol{u}(t)\|_{H_0^1(U)} + \|\boldsymbol{u}\|_{L^2(0,T;H^2(U))} + \|\boldsymbol{u}'\|_{L^2(0,T;L^2(U))} \le C \left(\|\boldsymbol{f}\|_{L^2(0,T;L^2(U))} + \|g\|_{H_0^1(U)} \right),$$

where the constant C depends only on U, T and the coefficients of L.

(ii) In addition, assume that $g \in H^2(U)$ and $\boldsymbol{f}' \in L^2(0, T; L^2(U))$. Then

$$\boldsymbol{u} \in L^\infty(0, T; H^2(U)), \quad \boldsymbol{u}' \in L^\infty(0, T; L^2(U)) \cap L^2(0, T; H_0^1(U)), \quad \text{and} \quad \boldsymbol{u}'' \in L^2(0, T; H^{-1}(U)).$$

Furthermore, we have the estimate

$$\begin{aligned} \operatorname*{ess\,sup}_{0 \le t \le T} \left(\|\boldsymbol{u}(t)\|_{H^2(U)} + \|\boldsymbol{u}'(t)\|_{L^2(U)} \right) + \|\boldsymbol{u}'\|_{L^2(0,T;H_0^1(U))} + \|\boldsymbol{u}''\|_{L^2(0,T;H^{-1}(U))} \\ \le C \left(\|\boldsymbol{f}\|_{H^1(0,T;L^2(U))} + \|g\|_{H^2(U)} \right), \end{aligned}$$

where the constant C depends only on U, T and the coefficients of L.

Proof. See Evans [1] Theorem 5 of §7.1.3. □

Theorem 6.21 (Higher-order regularity)**.** *Suppose that $\boldsymbol{u} \in L^2(0, T; H_0^1(U))$ with $\boldsymbol{u}' \in L^2(0, T; H^{-1}(U))$ is* *the weak solution of the initial/boundary-value problem*

$$\begin{cases} u_t + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0,T], \\ u = g & \text{on } U \times \{t=0\}. \end{cases}$$

Let $m \in \mathbb{N}_0$, *and assume that* $g \in H^{2m+1}(U)$ *and* $\frac{d^k \boldsymbol{f}}{dt^k} \in L^2(0,T; H^{2m-2k}(U))$ *for all* $k = 0, 1, \cdots, m$. *Suppose also the following* m^{th}*-order compatibility condition holds:*

$$g_0 := g \in H_0^1(U), \quad g_1 := \boldsymbol{f}(0) - Lg_0 \in H_0^1(U), \quad \cdots, \quad g_m := \frac{d^{m-1}\boldsymbol{f}}{dt^{m-1}}(0) - Lg_{m-1} \in H_0^1(U),$$

Then

$$\frac{d^k \boldsymbol{u}}{dt^k} \in L^2(0,T; H^{2m+2-2k}(U)), \quad k = 0, 1, \cdots, m+1.$$

Furthermore, we have the estimate

$$\sum_{k=0}^{m+1} \left\| \frac{d^k \boldsymbol{u}}{dt^k} \right\|_{L^2(0,T;H^{2m+2-2k}(U))} \le C \left(\sum_{k=0}^{m} \left\| \frac{d^k \boldsymbol{f}}{dt^k} \right\|_{L^2(0,T;H^{2m-2k}(U))} + \|g\|_{H^{2m+1}(U)} \right),$$

where the constant C *depends only on* m, U, T *and the coefficients of* L.

Proof. See Evans [1] Theorem 6 of §7.1.3. □

Theorem 6.22 (Infinite differentiability). *Suppose that* $\boldsymbol{u} \in L^2(0,T; H_0^1(U))$ *with* $\boldsymbol{u}' \in L^2(0,T; H^{-1}(U))$ *is the weak solution of the initial/boundary-value problem*

$$\begin{cases} u_t + Lu = f & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0,T], \\ u = g & \text{on } U \times \{t=0\}. \end{cases}$$

Assume that $g \in C^\infty(\overline{U})$ *and* $f \in C^\infty(\overline{U}_T)$, *and the following* m^{th}*-order compatibility condition holds for all* $m \in \mathbb{N}$. *Then* $u \in C^\infty(\overline{U}_T)$.

Proof. Apply induction and Theorem 6.21. □

6.4 Maximum Principles

In this section, we work with the uniformly parabolic operator of non-divergence form:

$$\frac{\partial}{\partial t} + L, \quad \text{where } Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i,x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu.$$

We assume that the coefficients a^{ij}, b^j, c $(i, j = 1, \cdots, n)$ are smooth, and we write $\Gamma_T = \overline{U}_T - U_T$ for the parabolic boundary of U_T.

6.4.1 Weak Maximum Principles

Theorem 6.23 (Weak maximum principle)**.** *Let U be a bounded open set. Assume that $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$u_t + Lu \leq 0 \quad \text{in } U_T. \tag{6.30}$$

If either (i) the zeroth-order coefficient of L is $c \equiv 0$ in U_T, or (ii) $\max_{\overline{U}_T} u = 0$, then

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u.$$

Proof. We first assume the strict inequality $u_t + Lu < 0$ in U_T, and there exists a point $(x_0, t_0) \in U_T$ where u attains its maximum over $\overline{U}_T$. Similar to the proof of Theorem 5.23, $Lu \geq 0$ at (x_0, t_0). If $0 < t_0 < T$, then $u_t = 0$ at (x_0, t_0). If $t_0 = T$, we have $u_t \geq 0$ at (x_0, t_0). In either case, $u_t + Lu \geq 0$ at (x_0, t_0), a contradiction.

In the general case that (6.30) holds, we define $u^\epsilon(x, t) = u(x, t) - \epsilon_t$ where $\epsilon > 0$. Then

$$u_t^\epsilon + Lu^\epsilon = u_t - \epsilon + Lu < 0 \quad \text{in } U_T.$$

By the previous result,

$$\max_{\overline{U}_T} u - \epsilon T \leq \max_{\overline{U}_T} u^\epsilon = \max_{\Gamma_T} u^\epsilon \leq \max_{\Gamma_T} u.$$

Letting $\epsilon \downarrow 0$ concludes the proof. □

Theorem 6.24 (Weak maximum principle)**.** *Let U be a bounded open set, and let the zeroth-order coefficient of L satisfy $c \geq 0$ in U_T. Assume that $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$u_t + Lu \leq 0 \quad \text{in } U_T.$$

Then

$$\max_{\overline{U}_T} u \leq \max_{\Gamma_T} u^+.$$

Proof. We first assume the strict inequality $u_t + Lu < 0$ in U_T, and there exists a point $(x_0, t_0) \in U_T$ where u attains a positive maximum over $\overline{U}_T$. Then $u_t + (Lu - cu) \geq 0$ at (x_0, t_0). Since $c \geq 0$ and $u(x_0, t_0) > 0$, we can derive the same contradiction $u_t + Lu \geq 0$ at (x_0, t_0) as before.

In the general case, we define $u^\epsilon(x, t) = u(x, t) - \epsilon_t$ where $\epsilon > 0$. Then $u_t^\epsilon + Lu^\epsilon < 0$ in U_T, and

$$\max_{\overline{U}_T} u - \epsilon T \leq \max_{\overline{U}_T} u^\epsilon = \max_{\Gamma_T} u^\epsilon \leq \max_{\Gamma_T} u.$$

Letting $\epsilon \downarrow 0$ concludes the proof. □

Remark. (i) Likewise, if $c \equiv 0$ in U_T and

$$u_t + Lu \geq 0 \quad \text{in } U_T,$$

we have

$$\min_{\overline{U}_T} u = \min_{\Gamma_T} u.$$

If we only require $c \geq 0$ in U_T, then

$$\min_{\overline{U}_T} u \geq -\max_{\Gamma_T} u^-.$$

(ii) In particular, if $c \geq 0$ and $u_t + Lu = 0$ in U_T, we have

$$\max_{\overline{U}_T} |u| = \max_{\Gamma_T} |u|.$$

(iii) In fact, our conclusion also holds for time-varying domains. To be specific, let $\Omega \subset \mathbb{R}^{n+1}$ be a region in $\mathbb{R}^n \times [0,T]$, where $\{x \in \mathbb{R}^n : (x,t) \in \Omega\}$ is a nonempty open set in $\mathbb{R}^n$ for each $t \in (0,T]$. Then all our conclusions apply to the domain Ω and the parabolic boundary $\mathscr{P}\Omega = \overline{\Omega} \backslash \Omega$.

Corollary 6.25 (Uniqueness for Cauchy-Dirchlet problem)**.** *Let U be a bounded open set, and let the zeroth-order coefficient of L be $c \geq 0$ in U_T. Let $h \in C(\partial U)$. The Cauchy-Dirichlet problem*

$$\begin{cases} u_t + Lu = f & \text{in } U_T, \\ u = h & \text{on } \partial U, \\ u = g & \text{on } U \times \{t = 0\} \end{cases} \tag{6.31}$$

has at most one solution in $C^{1,2}(U_T) \cap C(\overline{U}_T)$, i.e. there may be no solution or a unique solution but cannot be two or more solutions.

Proof. Let $u, \widetilde{u} \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ be two solutions of the Cauchy-Dirichlet problem (6.31). Then

$$\begin{cases} v_t + Lv = 0 & \text{in } U_T, \\ v = 0 & \text{on } \partial U, \end{cases}$$

where $v = u - \widetilde{u}$. By the weak maximum principle, $|v| \equiv 0$ in U_T. ☐

6.4.2 Strong Maximum Principles

For uniformly parabolic operator, we also have a strong maximum principle. If a subsolution u attains its maximum at some interior point, then u is contant at all earlier times.

Lemma 6.26. *Let U be a connected, bounded and open set. Assume that $u \in C^{2,1}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$u_t + Lu \leq 0 \quad \text{in } U_T, \tag{6.32}$$

and $u \leq 0$ on Γ_T. If there exists $(x^*, t^*) \in U_T$ such that $u(x^*, t^*) = 0$, then $u \equiv 0$ on U_{t_0}.

Proof. Step I. Let $\lambda > 0$ and $R > 0$. We define a function $\psi \in C^{2,1}(\mathbb{R}^n \times \mathbb{R})$ by

$$\psi(x,t) = e^{-\lambda t} \left(R^2 - |x|^2\right)_+^3.$$

In the cylinder $B(0,R)\times[0,T]$, we have

$$\begin{aligned}\psi_t+L\psi&=-\lambda e^{-\lambda t}\left(R^2-|x|^2\right)^3\\&\quad+e^{-\lambda t}\left(R^2-|x|^2\right)\left(-24\sum_{i,j=1}^n a^{ij}x_ix_j+6\sum_{i=1}^n(b^ix_i-a^{ii})\left(R^2-|x|^2\right)+c\left(R^2-|x|^2\right)^2\right)\\&\le e^{-\lambda t}\left(R^2-|x|^2\right)\left(-24\theta R^2+\left(6\sum_{i=1}^n(b^ix_i-a^{ii})+24\theta\right)\left(R^2-|x|^2\right)+(c-\lambda)\left(R^2-|x|^2\right)^2\right).\end{aligned}$$

By taking

$$\lambda=\|c\|_\infty+\frac{3}{8\theta R^2}\left(n\max_{1\le i\le n}\|a^{ii}\|_\infty+n\max_{1\le i\le n}\|b^i\|_\infty R+4\theta\right)^2,$$

we have $\psi_t+L\psi\le 0$ in $B(0,R)\times[0,T]$. Clearly, $\psi_t+L\psi=0$ when $|x|>R$. Since $\psi\in C^{2,1}(\mathbb{R}^n\times\mathbb{R})$, we conclude that $\psi_t+L\psi\le 0$ in $\mathbb{R}^n\times[0,T]$.

Step II. Let $\phi(x,t)=\psi(x-t\xi,t)$, where $\xi\in\mathbb{R}^n$. Then

$$\phi_t+L\phi=\psi_t-\xi^\top D_x\psi+L\psi.$$

We can replace the coefficient b of L with $b-\xi$ and take an appropriate $\lambda>0$ to obtain $\phi_t+L\phi=0$ in $\mathbb{R}^n\times[0,T]$. Also, we have $\phi>0$ in the infinite oblique cylinder $\{(x,t):0\le t\le T,\ |x-t\xi|<R\}$.

Step III. Assume that $u(x_0,t_0)<0$, where $(x_0,t_0)\in U_{t^*}$. We take $R>0$ so small that $B(x_0,R)\subset U$ and $u(x,t_0)>0$ for all $x\in B(x_0,R)$. Then there exists $\epsilon>0$ such that $\epsilon\psi(x-x_0,t_0)+u(x,t_0)\le 0$ in $B(x_0,R)$. Note that $u\le 0$ and $u_t+Lu\le 0$ in U_T, and $\psi=0$ on $\partial B(x_0,R)\times\mathbb{R}$. We then apply the weak maximum principle on $\epsilon\psi+u$ to obtain that $\epsilon\psi+u\le 0$, and $u<0$ in $B(x_0,R)\times(t_0,T]$.

Similarly, for any $\xi\in\mathbb{R}^n$, if the oblique cylinder $\{(x,t):t_0\le t\le T_0,\ |x-x_0-t\xi|<R\}$ lies in $\overline{U}_T$, where $t_0<T_0\le T$, we apply a similar statement to obtain $u<0$ in this cylinder. In particular, $u(x,T_0)<0$ for all $x\in B(x_0+T_0\xi,R)$. Therefore, if $B(x_0,\rho)\subset U$, we have $u<0$ in $B(x_0,\rho)\times(t_0,T]$.

Step IV. Finally, we use a chain of balls $B(x_i,\rho_i)\subset U,\ i=1,2,\cdots,k$ with $x_1=x_0$, $x_{i+1}\in B(x_i,\rho_i)$ and $x_k=x^*$ and apply the above result in each ball. Then $u<0$ in $B(x_i,\rho_i)\times(t_0,T]$ for each i, and in particular $u(x^*,t^*)<0$, a contradiction! Hence $u\equiv 0$ on U_{t^*}. ☐

Theorem 6.27 (Strong maximum principle)**.** *Let U be a connected, bounded and open set. Assume that $u\in C^{2,1}(U_T)\cap C(\overline{U}_T)$ satisfies*

$$u_t+Lu\le 0\quad\text{in } U_T,$$

and u attains its maximum over $\overline{U}_T$ at a point $(x^*,t^*)\in U_T$.

(i) If the zeroth-order coefficient $c\equiv 0$ in U, then u is a constant on U_{t_0}.

(ii) If the zeroth-order coefficient $c\ge 0$ in U, and $u(x^*,t^*)\ge 0$, then u is a constant on U_{t_0}.

Proof. Let $M=u(x^*,t^*)=\max_{\overline{U}_T}u$. We apply Lemma 6.26 on $u-M$. ☐

Remark. We also have the strong minimum principle:

Let U be a connected, bounded and open set. Assume that $u\in C^{2,1}(U_T)\cap C(\overline{U}_T)$ satisfies $u_t+Lu\ge 0$ in U_T, and u attains its minimum over $\overline{U}_T$ at a point $(x^*,t^*)\in U_T$.

(i) If the zeroth-order coefficient $c\equiv 0$ in U, then u is a constant on U_{t_0}.

(ii) If the zeroth-order coefficient $c\le 0$ in U, and $u(x^*,t^*)\ge 0$, then u is a constant on U_{t_0}.

6.5 Second-order Parabolic PDE Semigroup

In this section, we use the tool of semigroup theory to study the parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu = 0 & \text{in } U_T, \\ u = 0 & \text{on } \partial U \times [0,T], \\ u = g & \text{on } \partial U \times \{t = 0\}, \end{cases} \tag{6.33}$$

where L is a uniformly elliptic operator of the divergence form:

$$Lu = -\sum_{i,j=1}^{n} (a^{ij} u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu.$$

We further suppose that

- U is an open, bounded set with ∂U smooth, and
- the coefficients a^{ij}, b^i, c $(i,j = 1, \cdots, n)$ are smooth on $\overline{U}$ and not dependent on the time variable t.

We can reinterpret (6.33) as the flow determined by a semigroup on the Hilbert space $L^2(U)$. We set

$$\mathfrak{D}(A) := H_0^1(U) \cap H^2(U),$$

and define $Au = -Lu$ for $u \in \mathfrak{D}(A)$. Then $A : \mathfrak{D}(A) \to L^2(U)$ is an unbounded linear operator. Before we proceed, we recall the Hille-Yosida-Phillips theorem, which discusses the generation of a semigroup from an infinitesimal generator.

Theorem 6.28 (Hille-Yosida-Phillips)**.** *Let $A : \mathfrak{D}(A) \to X$ be a linear operator with a dense domain $\mathfrak{D}(A) \subset X$ on a real Banach space X. Assume that A has a closed graph. Fix $\omega \in \mathbb{R}$ and $M \geq 1$. Then the following are equivalent:*

(i) A is the infinitesimal generator of a strong continuous semigroup $(S_t)_{t \geq 0}$ that satisfies

$$\|S_t\| \leq M e^{\omega t} \quad \textit{for all } t \geq 0. \tag{6.34}$$

(ii) For every real number $\lambda > \omega$, the operator $\lambda \operatorname{Id} - A : \mathfrak{D}(A) \to X$ is bijective, and

$$\|(\lambda \operatorname{Id} - A)^{-k}\| \leq \frac{M}{(\lambda - \omega)^k}, \quad \textit{for all } \lambda > \omega \textit{ and } k \in \mathbb{N}. \tag{6.35}$$

In that case, the strongly continuous semigroup $(S_t)_{t \geq 0}$ generated by A is uniquely determined by the strong operator limit

$$S_t u = \lim_{\lambda \uparrow \infty} e^{-t\lambda} \sum_{k=0}^{\infty} \frac{t^k \lambda^{2k}}{k!} (\lambda \operatorname{Id} - A)^{-k} u, \quad u \in X, \ t \geq 0.$$

Semigroup theory provides an elegant method for constructing a solution to the initial/boundary-value problem (6.33). We recall the energy estimate in (5.11):

$$\beta \|u\|_{H_0^1(U)}^2 \leq B(u,u) + \gamma \|u\|_{L^2(U)}^2, \tag{6.36}$$

where B is the associated bilinear form

$$B(u,v) = \int_U \left(\sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \right) dx, \quad u, v \in H_0^1(U).$$

Theorem 6.29 (Second-order parabolic PDE as semigroups)**.** *The operator A generates a strongly continuous semigroup $(S_t)_{t\geq 0}$ on $L^2(U)$ such that*

$$\|S_t\| \leq e^{\gamma t} \quad \textit{for all } t \geq 0.$$

Proof. We prove the theorem by verify the hypotheses of the Hille-Yosida-Phillips Theorem in the case $\omega = \gamma$ and $M = 1$. It is clear that the domain $\mathfrak{D}(A) = H_0^1(U) \cap H^2(U)$ is dense in $L^2(U)$.

Step I. We prove that $A : \mathfrak{D}(A) \to L^2(U)$ has a closed graph in $L^2(U) \times L^2(U)$. Let $(u_k)_{k=1}^\infty \subset \mathfrak{D}(A)$ satisfy

$$u_k \to u \quad \text{and} \quad Au_k \to f \quad \text{in } L^2(U).$$

According to the regularity Theorem 5.20, there exists a constant C such that

$$\|u_k - u_l\|_{H^2(U)} \leq C\left(\|Au_k - Au_l\|_{L^2(U)} + \|u_k - u_l\|_{L^2(U)}\right).$$

Then $(u_k)_{k=1}^\infty$ is a Cauchy sequence in $H^2(U)$, which converges to some $u \in H^2(U)$. Hence $u \in \mathfrak{D}(A)$, and $Au_k \to Au$ in $L^2(U)$. Hence $f = Au$ and A has a closed graph.

Step II. Now we check that $\lambda \operatorname{Id} - A : \mathfrak{D}(A) \to L^2(U)$ is bijective for each $\lambda > \gamma$. By Theorem 5.9 (existence theory) and Theorem 5.20 (regularity theory), for each $\lambda \geq \gamma$, the boundary-value problem

$$\begin{cases} \lambda u - Au = f & \text{in } U, \\ u = 0 & \text{on } \partial U \end{cases} \tag{6.37}$$

has a unique weak solution $u \in H_0^1(U) \cap H^2(U)$, which satisfies $(\lambda \operatorname{Id} - A)u = f$. Hence the operator $\lambda \operatorname{Id} - A : \mathfrak{D}(A) \to L^2(U)$ is bijective, and $[\gamma, \infty) \subset \rho(A)$.

Step III. Finally we check the resolvent condition (6.35) with $\omega = \gamma$ and $M = 1$. Consider the weak formulation of the boundary-value problem (6.37):

$$B(u, v) + \lambda\langle u, v\rangle_{L^2(U)} = \langle f, v\rangle_{L^2(U)}, \quad \text{for each } v \in L^2(U).$$

Setting $\lambda > \gamma$, $v = u$ and applying the energy estimate (6.36), we have

$$(\lambda - \gamma)\|u\|_{L^2(U)}^2 \leq B(u, u) + \lambda\|u\|_{L^2(U)}^2 = \langle f, u\rangle_{L^2(U)} \leq \|f\|_{L^2(U)}\|u\|_{L^2(U)}.$$

Note that $u = (\lambda \operatorname{Id} - A)^{-1} f$. Then

$$\|(\lambda \operatorname{Id} - A)^{-1} f\|_{L^2(U)} \leq \frac{\|f\|_{L^2(U)}}{\lambda - \gamma}, \quad \text{for all } f \in L^2(U).$$

Therefore for each $\lambda > \gamma$, we have $\|(\lambda \operatorname{Id} - A)^{-1}\| \leq \frac{1}{\lambda - \gamma}$, and $\|(\lambda \operatorname{Id} - A)^{-k}\| \leq \frac{1}{(\lambda - \gamma)^k}$ for all $k \in \mathbb{N}$. Then we conclude the proof. □

Remark. We fix $g \in L^2(U)$, and let $(S_t)_{t\geq 0}$ be the strongly continuous semigroup generated by $A = -L$. Then the function $u(t, \cdot) = S_t g$ satisfies the following:

- for each $t > 0$, the function $u(t, \cdot) \in \mathfrak{D}(A) = H_0^1(U) \cap H^2(U)$;
- $u : U_T \to \mathbb{R}$ solves the linear Cauchy equation

$$\frac{\partial u}{\partial t} = Au, \quad u(0, \cdot) = g.$$

Hence u is a weak solution of the boundary/initial-value problem (6.33).

7 Calculus of Variations

7.1 Introduction

In this section, we introduce a new method of solving partial differential equations

$$A[u] = 0,$$

where $A[\cdot]$ is a possibly non-linear partial differential operator and u is the unknown. The calculus of variations identifies an important class of PDEs and transform the problem to an optimization problem. To be specific, we aim to find an appropriate *energy functional* $J[\cdot]$ whose "derivative" is $A[\cdot]$.

7.1.1 The Dirichlet Principle

We consider Poisson's equation

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases} \tag{7.1}$$

where U is a bounded, open subset of $\mathbb{R}^n$, $f \in C(U)$ and $g \in H^1(U)$. According to Corollary 5.25, the boundary-value problem (7.1) has at most one solution in $C^2(U) \cap C(\overline{U})$. We demonstrate that the solution can be characterized as the minimizer of an appropriate functional. Define the energe functional

$$J[w] = \int_U \left(\frac{1}{2}|Dw|^2 - wf \right) dx,$$

where w belongs to the *admissible set*

$$\mathcal{A} = \left\{ w \in C^2(U) \cap C(\overline{U}) : w = g \text{ on } \partial U \right\}.$$

Theorem 7.1 (Dirichlet's principle). *Assume $u \in C^2(U) \cap C(\overline{U})$ solves Poisson's equation* (7.1). *Then*

$$J[u] = \min_{w \in \mathcal{A}} J[w]. \tag{7.2}$$

Conversely, if $u \in \mathcal{A}$ satisfies (7.2)*, then u solves* (7.1)*.*

Proof. Step I. Given any $w \in C^2(U) \cap C(\overline{U})$, we multiply both sides of Possion's equation by $w - u$ and use integration by parts. Since $w - u = 0$ on ∂U, the boundary term is eliminated, and

$$\begin{aligned} 0 &= \int_U (w-u)(-\Delta u - f)\, dx \\ &= \int_U (Du \cdot Dw - wf)\, dx - \int_U \left(|Du|^2 - uf\right) dx. \end{aligned}$$

We then apply the estimate $Du \cdot Dw \le \frac{1}{2}|Du|^2 + \frac{1}{2}|Dw|^2$ to obtain

$$\int_U \left(\frac{1}{2}|Du|^2 - uf \right) dx \le \int_U \left(\frac{1}{2}|Dw|^2 - wf \right) dx.$$

Since $u \in \mathcal{A}$, we conclude (7.2).

Step II. Conversely, if (7.2) holds, we fix $\varphi \in C_c^\infty(U)$ and define

$$j(\tau) = J(u + \tau\varphi), \quad \tau \in \mathbb{R}.$$

Since $u+\tau\varphi \in \mathcal{A}$ for each $\tau \in \mathbb{R}$, and $j(\tau)$ is minimized by 0, we should have $j'(0) = 0$, provided the derivative exists. On the other hand, note that

$$j(\tau) = \int_U \left(|Du + \tau D\varphi|^2 - (u+\tau\varphi)f\right) dx = \int_U \left(\frac{1}{2}|Du|^2 + \tau Du \cdot D\varphi + \frac{\tau^2}{2}|Dv|^2 - (u+\tau\varphi)f\right) dx.$$

Hence

$$0 = j'(0) = \int_U (Du \cdot D\varphi - \varphi f)\, dx = \int_U (-\Delta u - f)\varphi\, dx,$$

which holds for all $\varphi \in C_c^\infty(U)$. Hence $-\Delta u = f$, and u solves (7.1). $\square$

Example 7.2 (Generalized Dirichlet's principle)**.** Consider the linear elliptic equation

$$-\sum_{i,j=1}^n \left(a^{ij} u_{x_i}\right)_{x_j} = f \quad \text{in } U. \tag{7.3}$$

The associated energy function is given by

$$J[w] = \int_U \left(\frac{1}{2}\sum_{i,j=1}^n a^{ij} w_{x_i} w_{x_j} - wf\right) dx.$$

Similar to Theorem 7.1, we can use integration by parts to show that a solution to (7.3) must be a minimizer of the energy function J. As we will discuss later, the uniform ellipticity ensures the existence of a minimizer.

Theorem 7.3 (Abstract Dirichlet's principle)**.** *Let H be a Hilbert space and $f \in H^*$. Define*

$$J[w] = \frac{1}{2}\|w\|_H^2 - f(w), \quad w \in H.$$

Then functional J has a unique minimizer u in H, and every minimizing sequence sequence converges to it, i.e., $J[u_k] \to \inf_{w\in H} J[w]$ implies $u_k \to u$ in H. Finally, the minimizer u is characterized by

$$0 = \frac{\partial}{\partial t} J[u+tw]\bigg|_{t=0} = \langle u, w\rangle_H - f(w) = 0 \quad \textit{for all } w \in H.$$

Proof. First note that $\inf_{w\in H} J[w] > -\infty$, since

$$J[w] = \frac{1}{2}\|w\|_H^2 - f(w) \geq \frac{1}{2}\|w\|_H^2 - \|f\|_{H^*}\|w\|_H \geq -\frac{1}{2}\|f\|_{H^*} > -\infty.$$

Next, we apply the parallelogram identity to obtain

$$\frac{1}{4}\|u-v\|_H^2 = \frac{1}{2}\|u\|_H^2 + \frac{1}{2}\|v\|_H^2 - \frac{1}{4}\|u+v\|_H^2 = J[u] + J[v] - 2J\left[\frac{u+v}{2}\right], \quad u, v \in H.$$

If both $u, v \in H$ are minimizers of J, we have $\frac{1}{4}\|u-v\|_H^2 = 2\inf_H J - 2J\left[\frac{u+v}{2}\right] \leq 0$, and $u = v$. This proves the uniqueness of minimizer. For the existence, we plug-in $u = u_k$ and $v = u_m$ to get

$$\frac{1}{4}\|u_k - u_m\|_H^2 \leq J[u_k] + J[u_m] - 2\inf_H J.$$

Hence any minimizing sequence is a Cauchy sequence, and the minimum exists by completeness of H. Finally, the characterization of minimizer u is obtained by forcing the derivative of $j(t) = J[u+tw]$ to vanish at $t = 0$, analogous to the proof of Theorem 7.1. $\square$

7.1.2 The Euler-Lagrange System

We consider the integral energy functional

$$J[w] = \int_U L(Dw, w, x)\,dx, \tag{7.4}$$

where U is a bounded, open set with smooth boundary ∂U, and $L : \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \overline{U} \to \mathbb{R}$ is a smooth function called *Lagrangian*. For clarity of our notation, we write

$$L = L(P, z, x) = L(p_1^1, \cdots, p_n^m, z^1, \cdots, z^m, x_1, \cdots, x_n)$$

for $P \in \mathbb{R}^{m\times n}$, $z \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$, where

$$P = \begin{pmatrix} p_1^1 & \cdots & p_n^1 \\ \vdots & \ddots & \vdots \\ p_1^m & \cdots & p_n^m \end{pmatrix}.$$

We write $L_{p_i^k}, L_{z^k}, L_{x_i}$ for the partial derivatives of L with respect to certain variables. For a differentiable function $w = (w^1, \cdots, w^m) : \mathbb{R}^n \to \mathbb{R}^m$, its gradient/Jacobian matrix is given by

$$Dw = \begin{pmatrix} w_{x_1}^1 & \cdots & w_{x_n}^1 \\ \vdots & \ddots & \vdots \\ w_{x_1}^m & \cdots & w_{x_n}^m \end{pmatrix}.$$

Connection to PDEs. Now we show that, to each minimization problem of the energy functional as above, one can associate a PDE. Let $u \in C^\infty(U; \mathbb{R}^m)$ be a smooth minimizer of $J[\cdot]$, taken among functions equal to a function $g : \mathbb{R}^n \to \mathbb{R}^m$ on ∂U, and fix any $\varphi \in C_c^\infty(U; \mathbb{R}^m)$. We define the *first variation*

$$j(\tau) = J[u + \tau\varphi] = \int_U L(Du + \tau D\varphi, u + \tau\varphi, x)\,dx, \quad \tau \in \mathbb{R}.$$

Then

$$j'(\tau) = \int_U \sum_{k=1}^m \sum_{i=1}^n \varphi_{x_i}^k L_{p_i^k}(Du + \tau D\varphi, u + \tau\varphi, x) + \sum_{k=1}^m \varphi^k L_{z^k}(Du + \tau D\varphi, u + \tau\varphi, x)\,dx.$$

Since u is a minimizer of J and $u + \tau\varphi$ is in the domain of J for all $\tau \in \mathbb{R}$,

$$0 = j'(0) = \sum_{k=1}^m \int_U \sum_{i=1}^n \varphi_{x_i}^k L_{p_i^k}(Du, u, x) + \varphi^k L_{z^k}(Du, u, x)\,dx.$$

As this identity holds for all $\varphi = (\varphi^1, \cdots, \varphi^m) \in C_c^\infty(U; \mathbb{R}^m)$, we apply integration by parts to conclude that

$$-\sum_{i=1}^n \Big(L_{p_i^k}(Du, u, x)\Big)_{x_i} + L_{z^k}(Du, u, x) = 0 \quad \text{in } U, \qquad k = 1, 2, \cdots, m.$$

This is known as the *Euler–Lagrange system* associated with the energy functional $J[\cdot]$ defined by (7.4), which is a coupled system of m quasilinear second-order PDEs. In particular, when $m = 1$, we write the Lagrangian $L = L(p, z) = L(p_1, \cdots, p_n, z, x)$ and obtain the *Euler-Lagrange equation*

$$-\sum_{i=1}^n (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) = 0.$$

Example 7.4. Following are some example of nonlinear PDEs associated with certain energy functionals.

(i) (Nonlinear Poisson equation). Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, and define its antilinear derivative $F(z) = \int_0^z f(t)\,dt$. Then the Euler-Lagrange equation associated with the energy functional

$$J[w] = \int_U \left(\frac{1}{2}|Dw|^2 - F(w) \right) dx$$

is the nonlinear Poisson equation

$$-\Delta u = f(u) \quad \text{in } U.$$

(ii) (Minimal surface equation). Define the Lagrangian

$$L(p, z, x) = \sqrt{1 + |p|^2}.$$

Then the energy functional is

$$J[w] = \int_U \sqrt{1 + |Dw|^2}\, dx,$$

and the associated Euler-Lagrange equation is the minimal surface equation

$$-\sum_{i=1}^{n} \left(\frac{u_{x_i}}{\sqrt{1 + |Du|^2}} \right)_{x_i} = 0 \quad \text{in } U.$$

The left hand side $\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}} \right)$ is n times the mean curvature of the graph of u.

Now we study certain systems of nonlinear PDEs for which every smooth function is a solution.

Definition 7.5 (Null Lagrangians)**.** The function L is called a *null Lagrangian* if every smooth function $u \in C^\infty(U; \mathbb{R}^m)$ solves the associated Euler-Lagrange system

$$-\sum_{i=1}^{n} \Big(L_{p_i^k}(Du, u, x) \Big)_{x_i} + L_{z^k}(Du, u, x) = 0 \quad \text{in } U, \qquad k = 1, 2, \cdots, m. \tag{7.5}$$

Remark. For the case $m = 1$, a trivial example of null Lagrangians is the linear function in p:

$$L(p, z, x) = \beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_n p_n.$$

The energy functional corresponding to a null Lagrangian only depends on the boundary condition.

Theorem 7.6 (Null Lagrangians and boundary conditions)**.** *Let L be a null Lagrangian, and*

$$J[w] = \int_U L(Dw, w, x)\, dx.$$

the corresponding energy functional. For any two functions $u, v \in C^\infty(\overline{U}; \mathbb{R}^m)$ with $u \equiv v$ on ∂U,

$$J[u] = J[v].$$

Proof. We define

$$j(\tau) = J[\tau u + (1 - \tau)v], \quad 0 \le \tau \le 1.$$

For notation simplicity we write $w_\tau = \tau u + (1-\tau)v$. Then

$$j'(\tau) = \int_U \sum_{k=1}^m \sum_{i=1}^n (u^k_{x_i} - v^k_{x_i}) L_{p^k_i}(Dw_\tau, w_\tau, x) + \sum_{k=1}^m (u^k - v^k) L_{z^k}(Dw_\tau, w_\tau, x)\, dx.$$

Since L is a null Lagrangian, w_τ solves (7.5), and

$$j'(\tau) = \sum_{k=1}^m \int_U (u^k - v^k) \left(- \sum_{i=1}^n \Big(L_{p^k_i}(Dw_\tau, w_\tau, x) \Big)_{x_i} + L_{z^k}(Dw_\tau, w_\tau, x) \right) dx = 0, \quad 0 \le \tau \le 1.$$

Hence $j(0) = j(1)$, and we finish the proof. $\square$

Now we introduce a nontrivial null Lagrangian.

Lemma 7.7 (Divergence-free rows). *Given a matrix $P \in \mathbb{R}^{n\times n}$, denote by $P^\sharp$ the cofactor matrix of P, whose $(k,i)^{th}$ entry is the cofactor*

$$(P^\sharp)^k_i = (-1)^{k+i} \det(A^{-k}_{-i}),$$

where P^{-k}_{-i} is the $(n-1)\times(n-1)$ matrix obtained by removing the k^{th} row and i^{th} column of P. Let $u \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be a smooth mapping. Then

$$\sum_{i=1}^n (Du^\sharp)^k_{i,x_i} = 0, \quad k = 1, 2, \cdots, n.$$

Proof. According to the identity $(\det P)\,\mathrm{Id} = P^T P^\sharp$, we have

$$(\det P)\delta_{ki} = \sum_{j=1}^n p^j_k (P^\sharp)^j_i, \quad k, i = 1, \cdots, n, \tag{7.6}$$

and in particular,

$$\frac{\partial \det P}{\partial p^m_j} = (P^\sharp)^m_j, \quad m, j = 1, \cdots, n. \tag{7.7}$$

We set $P = Du$ in (7.6) and differentiate with respect to x_i to obtain

$$\delta_{ki} \sum_{j,m=1}^n (Du^\sharp)^m_j (Du)^m_{j,x_i} = \sum_{j=1}^n \Big((Du)^j_{k,x_i} (Du^\sharp)^j_i + (Du)^j_k (Du^\sharp)^j_{i,x_i} \Big).$$

We then sum over all $i = 1, \cdots, n$ to obtain

$$\sum_{j,m=1}^n (Du^\sharp)^m_j u^m_{x_j x_k} = \sum_{i,j=1}^n \Big(u^j_{x_k x_i} (Du^\sharp)^j_i + (Du)^j_k (Du^\sharp)^j_{i,x_i} \Big), \quad k = 1, \cdots, n,$$

which also reads

$$\sum_{j=1}^n u^j_{x_k} \left(\sum_{i=1}^n (Du^\sharp)^j_{i,x_i} \right) = 0, \quad k = 1, \cdots, n.$$

If $\det Du(x) \neq 0$, we have $\sum_{i=1}^n (Du^\sharp)^j_{i,x_i} = 0$ at x for all $j = 1, \cdots, n$. Otherwise, if $\det Du(x) = 0$, we take $\widetilde{u}(y) = u(y) + \epsilon y$ for $\epsilon > 0$, apply the previous steps to $\widetilde{u}$ and send $\epsilon \downarrow 0$. $\square$

Theorem 7.8 (Determinants as null Lagrangians)**.** *The deterministic function*

$$L(P, z, x) = \det P$$

is a null Lagrangian.

Proof. Fix $u \in C^\infty(U; \mathbb{R}^m)$. By (7.7) and Lemma 7.7, we have

$$\sum_{i=1}^{n} \left(L_{p_i^k}(Du, u, x) \right)_{x_i} = \sum_{i=1}^{n} (Du^\sharp)^k_{i,x_i} = 0, \quad k = 1, \cdots, n.$$

Hence u solves the associated Euler-Lagrangian system, and we finish the proof. □

7.1.3 Application: Fixed Point Theorems

We can apply the null Lagrangians to provide an analytic proof for a fundamental result in algebraic topology.

Theorem 7.9 (Brouwer's fixed point theorem)**.** *Let $u : B \to B$ be a continuous mapping, where $B = \overline{B(0,1)}$ is the closed unit ball in $\mathbb{R}^n$. Then u has a fixed point, i.e. there exists $x \in B$ such that $u(x) = x$.*

Proof. Step I. We first claim that there does not exist a smooth mapping $w : B \to \partial B$ such that $w(x) = x$ for all $x \in \partial B$. Indeed, if w were such a mapping, then $w = \mathrm{Id}_B$ on ∂B. Since the determinant is a null Lagrangian, by Theorem 7.6, we have

$$\int_B \det Dw \, dx = \int_B \det D \, \mathrm{Id}_B \; dx = m(B) > 0. \tag{7.8}$$

On the other hand, since w takes values in ∂B, we have $|w|^2 = 1$, and the gradient $(Dw)^\top w = 0$. Note $|w| = 1$. Then 0 is an eigenvalue of Dw for each $x \in B$, and $\det Dw \equiv 0$ in B, contadicting (7.8).

Step II. We then claim that there does not exist a continuous mapping $w : B \to \partial B$ such that $w(x) = x$ for all $x \in \partial B$. If w were such a mapping, we continuously extend to $\mathbb{R}^n$ by assigning $w(x) = x$ for all $x \in \mathbb{R}^n \backslash B$. We take $w^\epsilon = \eta_\epsilon * w$, where $0 < \epsilon < 1$ and $\eta_\epsilon \in C_c^\infty(\mathbb{R}^n)$ is a standard mollifier. By Proposition 1.8 (ii), $w^\epsilon \to w$ uniformly on $\overline{B(0,2)}$ as $\epsilon \downarrow 0$. On the other hand, since η_ϵ is radial, $w^\epsilon(x) = x$ for all $x \in \mathbb{R}^n \backslash B(0,2)$. Consequently, we fix $\epsilon \in (0,1)$ so small that $\inf_{x \in \mathbb{R}^n} |w^\epsilon(x)| > 0$. Then

$$w_2(x) = \frac{w^\epsilon(2x)}{|w^\epsilon(2x)|}, \quad x \in B$$

is a smooth mapping from B to ∂B such that $w_2(x) = x$ for all $x \in \partial B$, contradicting our claim in Step I.

Step III. Finally, assume that $u : B \to B$ is a continuous mapping. If u has no fixed point, we define the mapping $w : B \to \partial B$ by setting $w(x)$ to be the point on ∂B hit by the ray emanating from $u(x)$ and passing through x. Since $u(x) \neq x$ for each $x \in B$, this mapping is well-defined. To be explicit,

$$w(x) = x + \tau(x)(u(x) - x), \quad x \in B,$$

where

$$\tau(x) = \frac{x^\top (x - u(x)) - \sqrt{(1 - |x|^2) \, |u(x) - x|^2 + |x^\top (x - u(x))|^2}}{|u(x) - x|^2}.$$

Therfore, $w : B \to \partial B$ is a continuous mapping and satisfies $w(x) = x$ for all $x \in \partial B$, which contradicts our claim in Step II. Then we complete the proof. □

The Brouwer's fixed point theorem can be easily generalize to homeomorphic spaces of the closed unit ball in Euclidean spaces.

Corollary 7.10 (Brouwer's fixed point theorem)**.** *Let A be a topological space that is homeomorphic to the closed unit ball B in $\mathbb{R}^n$, i.e. there exists a bijection $f : A \to B$ such that both f and f^{-1} are continuous. Then every continuous mapping $u : A \to A$ also has a fixed point.*

Proof. We note that the mapping $f \circ u \circ f^{-1} : B \to B$ is also a continuous mapping, which has a fixed point $x \in B$ by Theorem 7.9. Clearly, $f^{-1}(x) \in A$ is a fixed point of u. □

We can further generalize Brouwer's fixed point theorem to Banach spaces.

Theorem 7.11 (Schauder's fixed point theorem)**.** *Let X be a Banach space. If $K \subset X$ is a compact and convex subset, and $T : X \to X$ is a continuous mapping, then T has a fixed point in K.*

Proof. Step I. By compactness of K, we fix $\epsilon > 0$ and cover K by finitely open balls $B(x_1, \epsilon), \cdots, B(x_{N_\epsilon}, \epsilon)$, where $x_1, \cdots, x_{N_\epsilon} \in K$. We take K_ϵ to be the convex hull of these centers. Since K is convex, $K_\epsilon \subset K$.

Step II. We claim that K_ϵ is homeomorphic to the closed unit ball in $\mathbb{R}^{M_\epsilon}$ for some $M_\epsilon \le N_\epsilon - 1$. Without loss of generality, we assume that $0 \in K_\epsilon$, and take M_ϵ to be the dimension of the subspace spanned by $\{x_1, \cdots, x_{N_\epsilon}\}$. Then K_ϵ lies in a M_ϵ-dimensional real vector space, which is homeomorphic to R^{M_ϵ}.

- If $M_\epsilon = 0$, then K_ϵ is a singleton, and the result is clear;
- If $M_\epsilon \ge 1$, then K_ϵ is a compact and convex set with nonempty interior in a M_ϵ-dimensional real vector space, which is homeomorhic to the closed unit ball in $\mathbb{R}^{M_\epsilon}$.

Step III. We define a mapping $S_\epsilon : K \to K_\epsilon$ by

$$S_\epsilon x = \frac{\sum_{j=1}^{N_\epsilon} d(x, K\backslash B(x_j, \epsilon)) x_j}{\sum_{j=1}^{N_\epsilon} d(x, K\backslash B(x_j, \epsilon))}, \quad x \in K,$$

where the denominator is never zero because $B(x_1, \epsilon), \cdots, B(x_{N_\epsilon}, \epsilon)$ cover K. Clearly, S_ϵ is a continuous mapping. Furthermore, since $d(x, K\backslash B(x_j, \epsilon)) > 0$ if and only if $x \in B(x_j, \epsilon)$, we have the estimate

$$\|S_\epsilon x - x\| \le \frac{\sum_{j=1}^{N_\epsilon} d(x, K\backslash B(x_j, \epsilon)) \|x - x_j\|}{\sum_{j=1}^{N_\epsilon} d(x, K\backslash B(x_j, \epsilon))} \le \epsilon, \quad x \in K. \tag{7.9}$$

Step IV. We further define a mapping $T_\epsilon : K_\epsilon \to K_\epsilon$ by

$$T_\epsilon x = S_\epsilon(Tx), \quad x \in K_\epsilon,$$

which is also continuous. Since K_ϵ is homeomorphic to the closed unit ball in $\mathbb{R}^{M_\epsilon}$, by Brouwer's fixed point theorem, there exists $x_\epsilon \in K_\epsilon$ such that $T_\epsilon x_\epsilon = x_\epsilon$. Since K is compact, there exists a subsequence ϵ_j such that x_{ϵ_j} converges to a limit $x \in X$. By estimate (7.9),

$$\|x_{\epsilon_j} - Tx_{\epsilon_j}\| \le \|T_{\epsilon_j} x_{\epsilon_j} - Tx_{\epsilon_j}\| \le \|S_{\epsilon_j} Tx_{\epsilon_j} - Tx_{\epsilon_j}\| \le \epsilon_j.$$

Since T is continuous, $\|x_{\epsilon_j} - Tx_{\epsilon_j}\| \to \|x - Tx\| = 0$. Hence we conclude $Tx = x$. □

7.2 Existence Theory for Variational Problems

In this section, we discuss some conditions on the Lagrangian $L : \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \overline{U} \to \mathbb{R}$ which ensure that the energy functional $J[\cdot]$ defined by

$$J[w] = \int_U L(Dw, w, x)\, dx$$

indeed has a minimizer, where $w : U \to \mathbb{R}^m$ is taken over an appropriate Sobolev space, possibly under some boundary conditions.

7.2.1 Existence and Uniqueness of Minimizers

Some functions have an infimum but do not have a minimizer, for instance, the functions that vanish at infinity, like $e^{-|x|^2}$. Heurestically, we may require hypothesis that controls the value of the objective functional for points near infinity. Also, to ensure that the functional attains its infimum, we need some continuity conditions. In fact, there is a systematic approach for constructing minimizers, which is based on the so-called Direct method of the calculus of variations.

Theorem 7.12 (Direct method)**.** *Let X be a reflexive Banach space, and let $A \subset X$ be a weakly closed subset. Let $J : A \to \mathbb{R}$ be a (possibly nonlinear) functional satisfying the following conditions:*

(i) (Coercivity). $J[u] \to \infty$ as $\|u\| \to \infty$, and

(ii) (Sequential weak lower semicontinuity). If $(u_k) \subset A$ and $u_k \to u$ weakly in u, then

$$J[u] \le \liminf_{k\to\infty} J[u_k].$$

Then J is bounded from below on A and attains its infimum on A.

Proof. Let (u_k) be a minimizing sequence in A, i.e.

$$\lim_{k\to\infty} J[u_k] = \inf_{v\in A} J[v].$$

By the coercivity condition, (u_k) is a bounded sequence, which has a weakly convergent subsequence (u_{k_j}) because X is reflexive. Since A is weakly closed, the weak limit u is in A. By weak lower semicontinuity,

$$J[u] \le \liminf_{k\to\infty} J[u_k] = \inf_{v\in A} J[v].$$

Therefore J attains its infimum on A at u. Since $J[u] > -\infty$, the conclusion follows. □

Coercivity. In accordance with the coercivity condition in the direct method, we hope the energy functional $J[w]$ to grow rapidly as w tends to infinity. To this end, we assume that for some $1 < q < \infty$, there exist constants $\alpha > 0, \beta \ge 0$ such that

$$L(P, z, x) \ge \alpha|P|^q - \beta \quad \text{for all } (P, z, x) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \overline{U}.$$

Therefore

$$J[w] \ge \delta\|Dw\|^q_{L^q(U)} - \gamma$$

for $\gamma = \beta m(U)$ and some $\delta > 0$, and $J[w] \to \infty$ as $\|Dw\|_{L^q(U)} \to \infty$.

Weak convergence in $W^{1,q}(U)$. Since we assume $1 < q < \infty$, the space $L^q(U)$ is a reflexive space. For a bounded sequence (u_k) in $W^{1,q}(U)$, there exists a subsequence (u_{k_j}) and a function $u \in W^{1,q}(U)$ such that $u_{k_j} \to u$ weakly in $L^q(U)$ and $Du_{k_j} \to Du$ weakly in $L^q(U; \mathbb{R}^n)$. For brevity, we say $u_{k_j} \to u$ in $W^{1,q}(U)$.

Now we discuss the relation between convexity and weak lower semicontinuity.

Definition 7.13 (Tonelli)**.** Assume that $L : \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \overline{U} \to \mathbb{R}$ is smooth and bounded from below, and the mapping $P \mapsto L(P, z, x)$ is convex for every $(z, x) \in \mathbb{R}^m \times \overline{U}$. Then $J[\cdot]$ is sequentially weakly lower semicontinuous on $W^{1,q}(U;\mathbb{R}^m)$ for each $q \in (1,\infty)$.

Proof. We may assume $L \geq 0$, otherwise we add a large constant to L since it is bounded from below. Let (u_k) be a weakly convergent sequence in $W^{1,q}(U;\mathbb{R}^m)$ with weak limit u. We aim to show that

$$J[u] \leq \liminf_{k\to\infty} J[u_k].$$

By passing to an appropriate subsequence, we may replace the lim inf with an actual limit. Since (u_k) is weakly convergent in $W^{1,q}(U;\mathbb{R}^m)$, it is bounded. By Rellich-Kondrachov compactness theorem [Theorem 4.16], we have $u_k \to u$ in $L^q(U;\mathbb{R}^n)$ up to a subsequence, and $u_k \to u$ a.e. up to a further subsequence. Now we fix $\epsilon > 0$ and apply Egoroff's therorem to conclude that there exists a set $E_\epsilon \subset U$ such that $m(U\backslash E_\epsilon) < \epsilon$ and $u_k \to u$ uniformly on E_ϵ. We may assume $E_\epsilon \subset E_{\epsilon'}$ for $0 < \epsilon' < \epsilon$. We take the good set

$$G_\epsilon = \left\{ x \in E_\epsilon : |u(x)| + |Du(x)| < \frac{1}{\epsilon} \right\},$$

so $m(U\backslash G_\epsilon) \downarrow 0$ as $\epsilon \downarrow 0$.

By convexity of $L(P, z, x)$ in P and the fact $L \geq 0$, we have

$$J[u_k] \geq \int_{G_\epsilon} L(Du_k, u_k, x)\, dx \geq \int_{G_\epsilon} L(Du, u_k, x)\, dx + \int_{G_\epsilon} L_P(Du, u_k, x)\cdot(Du_k - Du)\, dx. \tag{7.10}$$

By definition of G_ϵ and the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{G_\epsilon} L(Du, u_k, x)\, dx = \int_{G_\epsilon} L(Du, u, x)\, dx.$$

Also, since $L_P(Du, u_k, x) \to L_P(Du, u, x)$ uniformly on G_ϵ, and $Du_k \to Du$ weakly in $L^q(\overline{U};\mathbb{R}^m)$, we have

$$\lim_{k\to\infty} \int_{G_\epsilon} L_P(Du, u_k, x)\cdot(Du_k - Du)\, dx = 0.$$

Hence for each $\epsilon > 0$, we let $n \to \infty$ in (7.10) to see

$$\lim_{k\to\infty} J[u_k] \geq \int_{G_\epsilon} L(Du, u, x)\, dx.$$

We let $\epsilon \downarrow 0$ and apply the monotone convergence theorem to conclude the proof. □

Minimizers for the variational problem. We discuss the variational problem under Dirichlet boundary conditions, where the energy functional

$$J[w] = \int_U L(Dw, w, x)\, dx$$

is defined on the *admissible set*

$$\mathcal{A} = \left\{ w \in W^{1,q}(U;\mathbb{R}^m) : w = g \text{ on } \partial U \text{ in the trace sense} \right\}.$$

We say $u \in W^{1,q}(U;\mathbb{R}^m)$ *is a minimizer of J, if $J[u] \leq J[u+\varphi]$ for all $\varphi \in W_0^{1,q}(U;\mathbb{R}^m)$*. In other words, the Sobolev function u minimizes J in its own Dirichlet class $W_u^{1,q}(U;\mathbb{R}^m) := u + W_0^{1,q}(U;\mathbb{R}^m)$.

Theorem 7.14 (Existence of Minimizers)**.** *Let $1 < q < \infty$. Assume that $L : \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \overline{U} \to \mathbb{R}$ is smooth and bounded from below, the mapping $P \mapsto L(P, z, x)$ is convex for every $(z, x) \in \mathbb{R}^m \times \overline{U}$, and there exist constants $\alpha > 0, \beta \geq 0$ such that*

$$L(P, z, x) \geq \alpha|P|^q - \beta \quad \textit{for all } (P, z, x) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \overline{U}.$$

Then for each $\varphi \in C^\infty(U;\mathbb{R}^m)$, the enregy functional J has a minimizer in $\mathcal{A} = \varphi + W_0^{1,q}(U;\mathbb{R}^m)$.

Proof. We first check the coercivity of J. We have shown that

$$J[w] \geq \|Dw\|^q_{L^q(U)} - \beta m(U)$$

for some $\delta > 0$. Since $w - \varphi \in W_0^{1,q}(U)$ for all $w \in \mathcal{A}$, by Poincaré's inequality [Corollary 4.6],

$$\|w\|_{L^q(U)} \leq \|w-\varphi\|_{L^q(U)} + \|\varphi\|_{L^q(U)} \leq C_1\|Dw - D\varphi\|_{L^q(U)} + \|\varphi\|_{L^q(U)} \leq C_2\left(\|Dw\|_{L^q(U)} + \|\varphi\|_{W^{1,q}(U)}\right).$$

Hence $\|w\|_{W^{1,q}(U)} \leq C_3(\|Dw\|_{L^q(U)} + 1)$. As $\|w\|_{W^{1,q}(U)} \to \infty$, we have $|Du\|_{L^q(U)} \to \infty$ and $J[w] \to \infty$.

Next, since $1 < q < \infty$, we know that $W^{1,q}(U)$ is a refelxive Banach space. Also, by Mazur's theorem, the admissible set $\mathcal{A} = v + W_0^{1,q}(U)$ is a weakly closed space. By Theorem 7.12, J attains its infimum on $\mathcal{A}$. □

Now we discuss the uniqueness of the minimizer.

Theorem 7.15 (Uniqueness)**.** *Assume that the Lagrangian $L = L(P, x)$ does not depend on z, and L is uniformly convex in P, i.e. there exists $\theta > 0$ such that*

$$\sum_{i,j=1}^n \sum_{k,l=1}^m L_{p_i^k p_j^l}(P, x)\xi_i^k \xi_j^l \geq \theta|\xi|^2$$

for all $P, \xi \in \mathbb{R}^{m\times n}$ and $x \in \mathbb{R}^n$. Then the minimizer $u \in \mathcal{A}$ of $J[\cdot]$ in Theorem 7.14 is unique.

Proof. Assume both $u, v \in \mathcal{A}$ minimizes $J[\cdot]$ over $\mathcal{A}$. By uniform convexity,

$$L(P, x) \geq L(Q, x) + D_P L(Q, x) \cdot (P - Q) + \frac{\theta}{2}|P - Q|^2, \quad x \in \mathbb{R}^n,\ P, Q \in \mathbb{R}^{m\times n}.$$

We set $P = Du$, $Q = \frac{Du+Dv}{2}$ and integrate over U:

$$J(u) \geq J\left(\frac{u+v}{2}\right) + \int_U D_P L\left(\frac{Du+Dv}{2}, x\right) \cdot \frac{Du - Dv}{2}\,dx + \frac{\theta}{8}\int_U |Du - Dv|^2\,dx.$$

Symmetrically,

$$J(v) \geq J\left(\frac{u+v}{2}\right) + \int_U D_P L\left(\frac{Du+Dv}{2}, x\right) \cdot \frac{Dv - Du}{2}\,dx + \frac{\theta}{8}\int_U |Dv - Du|^2\,dx.$$

Combining the last two displays, we have

$$J\left(\frac{u+v}{2}\right) + \frac{\theta}{8}\int_U |Dv - Du|^2\,dx \leq \frac{J(u) + J(v)}{2}.$$

Since both u and v are minimizers of $J[\cdot]$ over $\mathcal{A}$, and $\frac{u+v}{2} \in \mathcal{A}$, the above inequality is indeed an equality, and $Du = Dv$ a.e.. Since $u = v = \varphi$ on ∂U in the trace sense, it follows that $u = v$ a.e.. □

7.2.2 Weak Solutions of Euler-Lagrange System

In this section, we show that the minimizer of the energy functional $J[\cdot]$ on the admissible set $\mathcal{A}$ solves the associated Euler-Lagrange system in some suitable sense. Suppose there exists some constant C such that

$$\begin{cases} |L(P,z,x)| \le C\left(|P|^q + |z|^q + 1\right), \\ |L_P(P,z,x)| \le C\left(|P|^{q-1} + |z|^{q-1} + 1\right), \\ |L_z(P,z,x)| \le C\left(|P|^{q-1} + |z|^{q-1} + 1\right). \end{cases} \tag{7.11}$$

for all $P \in \mathbb{R}^{m\times n}$, $z \in \mathbb{R}^m$ and $x \in U$. Then for each $w \in W^{1,q}(U;\mathbb{R}^m)$, we have

$$C\left(|Dw|^{q-1} + |w|^{q-1} + 1\right) \in L^{q'}(U)$$

where q' is the conjugate $q' = \frac{q}{q-1}$. Consequently, both $|L_P(Dw,w,x)|$ and $|L_z(Dw,w,x)|$ are in $L^{q'}(U)$.

Weak formulation. Recall the Euler-Lagrange system associated with $J[w] = \int_U L(Dw,w,x)\,dx$:

$$\begin{cases} -\sum_{i=1}^n \left(L_{p_i^k}(Du,u,x)\right)_{x_i} + L_{z^k}(Du,u,x) = 0 & \text{in } U, \quad k = 1,\cdots,m, \\ u = g & \text{on } \partial U. \end{cases} \tag{7.12}$$

We multiply this system by a test function $v = (v^1,\cdots,v^m) \in C_c^\infty(U;\mathbb{R}^m)$ and integrate by parts to obtain

$$\int_U \left(\sum_{k=1}^m L_{p_i^k}(Du,u,x)v^k_{x_i}\,dx + L_{z^k}(Du,u,x)v^k\right)dx = 0, \quad k = 1,\cdots,m.$$

Consequently, we see using a standard approximation argument that the above equality remains valid for all $v \in W^{1,q}(U;\mathbb{R}^m)$. This motivates the following weak formulation.

Definition 7.16 (Weak solutions)**.** Let $u \in W_g^{1,q}(U;\mathbb{R}^m)$. Then u is said to be a *weak solution* of the Euler-Lagrange system (7.12) provided

$$\sum_{k=1}^m \int_U \left(\sum_{i=1}^n L_{p_i^k}(Du,u,x)v^k_{x_i}\,dx + L_{z^k}(Du,u,x)v^k\right)dx = 0 \tag{7.13}$$

for all $v = (v^1,\cdots,v^m) \in W_0^{1,q}(U;\mathbb{R}^m)$.

Accordingly, the minimizer of the energy functional solves the Euler-Lagrange system in the weak sense.

Theorem 7.17. *Suppose that the Lagrangian $L : \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \overline{U} \to \mathbb{R}$ satisfy the growth condition* (7.11)*, and $u \in W_g^{1,q}(U;\mathbb{R}^m)$ satisfies*

$$J[u] = \min_{w\in\mathcal{A}} J[w].$$

Then u is a weak solution of the Euler-Lagrange system (7.12)*.*

Proof. Without loss of generality, we proceed with $m = 1$. We fix $v \in W_0^{1,q}(U)$, and define

$$j(\tau) = J[u + \tau v], \quad \tau \in \mathbb{R}.$$

By (7.11), $|j(\tau)| < \infty$ for all $\tau \in \mathbb{R}$. We write the difference quotient

$$\frac{j(\tau) - j(0)}{\tau} = \int_U L^\tau(x)\,dx,$$

where

$$L^\tau(x) = \frac{1}{\tau}\left[L(Du(x)+\tau Dv(x), u(x)+\tau(x)v, x) - L(Du(x), u(x), x)\right] \quad \text{for a.e. } x \in U.$$

Note that

$$\begin{aligned} L^\tau(x) &= \frac{1}{\tau}\int_0^\tau \frac{d}{ds} L(Du+sDv, u+sv, x)\, ds \\ &= \frac{1}{\tau}\int_0^\tau Dv \cdot L_P(Du+sDv, u+sv, x)\, ds + \frac{1}{\tau}\int_0^\tau v \cdot L_z(Du+sDv, u+sv, x)\, ds. \end{aligned}$$

Since $u, v \in W^{1,q}(U;\mathbb{R}^m)$, by Young's inequality $ab \le \frac{a^q}{q} + \frac{b^{q'}}{q'}$ and the growth condition (7.11), we have

$$\begin{aligned} L^\tau(x) &\le \frac{1}{\tau}\int_0^\tau \left(\frac{|Dv|^q}{q} + \frac{|L_P(Du+sDv, u+sv, x)|^{q'}}{q'} + \frac{|v|^q}{q} + \frac{|L_z(Du+sDv, u+sv, x)|^{q'}}{q'} \right) ds \\ &\le \frac{1}{\tau}\int_0^\tau \left(\frac{|Dv|^q}{q} + \frac{|v|^q}{q} + \frac{C_1\left(|Du+sDv|^q + |u+sv|^q + 1\right)}{q'} \right) ds \\ &\le C_2\left(|Dv|^q + |v|^q + |Du|^q + |u|^q + 1\right) \end{aligned}$$

for some constant $C_1, C_2 > 0$ and all $0 < |\tau| \le 1$. Also, we let $\tau \to 0$ to get

$$\lim_{\tau\to 0} L^\tau(x) = Dv \cdot L_P(Du, u, x) + v \cdot L_z(Du, u, x) \quad \text{for a.e. } x \in U.$$

By the dominated convergence theorem, j is differentiable and

$$j'(0) = \lim_{\tau\to 0}\int_U L^\tau(x)\, dx = \int_U \left[Dv \cdot L_P(Du, u, x) + v \cdot L_z(Du, u, x)\right] dx,$$

Since j attains its minimum at $\tau = 0$, we have $j'(0) = 0$. Hence

$$0 = \int_U \left[Dv \cdot L_P(Du, u, x) + v \cdot L_z(Du, u, x)\right] dx = \sum_{k=1}^m \int_U \left(\sum_{i=1}^n L_{p_i^k}(Du, u, x) v^k_{x_i} + L_{z^k}(Du, u, x) v^k \right) dx,$$

and u is a weak solution of (7.12). □

In general, the minimizers of the energy functional do not capture all weak solutions of the Euler-Langrange system. Nevertheless, in the special case that the joint mapping $(P, z) \to L(P, z, x)$ is convex for each $x \in U$, then each weak solution is in fact a minimizer.

Proposition 7.18. *Assume that $(P, z) \mapsto L(P, z, x)$ is convex for each $x \in U$. Then each weak solution $u \in W_g^{1,q}(U;\mathbb{R}^m)$ of (7.12) is a minimizer of J over $W_g^{1,p}(U;\mathbb{R}^m)$.*

Proof. Suppose $u \in W_g^{1,q}(U;\mathbb{R}^m)$ solves (7.5) in the weak sense. For each $w \in W_g^{1,q}(U;\mathbb{R}^m)$, by convexity of the mapping $(P, z) \to L(P, z, x)$,

$$L(P, z, x) + L_P(P, z, x)\cdot(Q - P) + L_z(P, z, x)\cdot(y - z) \le L(Q, y, x).$$

We let $P = Du, Q = Dw, z = u$ and $y = w$, and integrate on U. Then

$$J[u] + \int_U \left[L_P(Du, u, x)\cdot(Dw - Du) + L_z(Du, u, x)\cdot(w - u)\right] dx \le J[w].$$

Since $v = w - u \in W_0^{1,q}(U;\mathbb{R}^m)$, the second term on the left side is zero. Hence $J[u] \le J[w]$. □

7.2.3 Local Minimizers

A cirtical point u of the energy functional

$$I[w] = \int_U L(Dw, w, x)\,dx$$

among functions w satisfying the boundary condition $w = g$ on ∂U satisfies the Euler-Lagrange system

$$\begin{cases} -\sum_{i=1}^n \left(L_{p_i^k}(Du, u, x)\right)_{x_i} + L_{z^k}(Du, u, x) = 0 & \text{in } U, \quad k = 1, \cdots, m, \\ u = g & \text{on } \partial U. \end{cases}$$

We assume u is a smooth solution of the Euler-Lagrangian PDE. In this section, we identify the case that u is a local minizer of the energy functional. For simplicity, we as usual let $P \mapsto L(P, z, x)$ be a convex mapping, and assume the graph of $x \mapsto u(x)$ lies within a region R generated by a parameteric family of graphs $x \mapsto u(x, \lambda)$ corresponding to other critical points. To be specific, we suppose $D \subset \mathbb{R}^m$ is an open set containing 0 and $\{u(\cdot, \lambda) : \lambda \in D\}$ is a smooth family of solutions of the Euler-Lagrange PDE

$$-\sum_{i=1}^n \left(L_{p_i^k}(Du(x,\lambda), u(x,\lambda), x)\right)_{x_i} + L_{z^k}(Du(x,\lambda), u(x,\lambda), x) = 0 \text{ in } U, \quad k = 1, \cdots, m, \tag{7.14}$$

with $u(x) = u(x, 0)$ in U. For notation simplicty, we write

$$A^{ki}(x, \lambda) = L_{p_i^k}\left(D_x u(x,\lambda), u(x,\lambda), x\right),$$

so (7.14) becomes

$$-\sum_{i=1}^n A^{ki}_{x_i}(x, \lambda) + L_{z^k}(Du(x,\lambda), u(x,\lambda), x) = 0 \text{ in } U, \quad k = 1, \cdots, m. \tag{7.15}$$

Next, we construct a subregion of admissible set by taking smooth functions $\theta : \overline{U} \to D$ with $\theta \equiv 0$ on ∂U, and defining

$$w(x) = u(x, \theta(x)), \quad x \in \mathbb{R}.$$

This function satisfies $w \equiv u = g$ on ∂U. We let $R \subset \mathcal{A}$ be the set of all functions w constructed as above.

Theorem 7.19. *The function $u = u(\cdot, 0)$ is a local minimizer within the region R, in the sense that*

$$I[u] \leq I[w]$$

for all functions w constructed as above.

Proof. Step I. Using the chain rule, the derivatives of $w(x) = u(x, \theta(x))$ is given by

$$w_{x_i}(x) = u_{x_i}(x, \theta(x)) + u_\lambda(x, \theta(x))\theta_{x_i}.$$

Then we use convexity of $P \mapsto L(P, z, x)$ to conclude

$$\begin{aligned} I[w] &= \int_U L(Dw, w, x)\,dx = \int_U L(D_x u(x,\theta) + u_\lambda(x,\theta)D\theta, w, x)\,dx \\ &\geq \int_U \left[L(D_x u(x,\theta), w, x) + L_P(D_x u(x,\theta), w, x) \cdot u_\lambda(x,\theta)D\theta\right] dx. \end{aligned} \tag{7.16}$$

Step II. We define a vector field $b = (b^1, \cdots, b^n) : \mathbb{R}^n \to \mathbb{R}^n$ by

$$b^i = \sum_{k=1}^m \int_0^1 A^{ki}(x, \theta)\, u_\lambda^k(x, t\theta) \cdot \theta \, dt, \quad x \in U.$$

Now we compute the divergence of b. By definition,

$$\begin{aligned}
\operatorname{div} b(x) &= \sum_{k=1}^m \sum_{i=1}^n \int_0^1 A^{ki}(x, t\theta) \left(u_\lambda^k(x, t\theta) \cdot \theta_{x_i} + u_{\lambda x_i}^k(x, t\theta) \cdot \theta + t u_{\lambda\lambda}^k(x, t\theta)\, \theta_{x_i} \cdot \theta \right) dt \\
&\quad + \sum_{k=1}^m \sum_{i=1}^n \int_0^1 \left(A_{x_i}^{ki}(x, t\theta) + A_\lambda^{ki}(x, t\theta) \cdot t\theta_{x_i} \right) u_\lambda^k(x, t\theta) \cdot \theta \, dt \\
&= \sum_{k=1}^m \sum_{i=1}^n \int_0^1 \left(A^{ki}(x, t\theta) u_\lambda^k(x, t\theta) + t A^{ki}(x, t\theta) u_{\lambda\lambda}^k(x, t\theta)\, \theta + \left(u_\lambda^k(x, t\theta) \cdot t\theta \right) A_\lambda^{ki}(x, t\theta) \right) \cdot \theta_{x_i} \, dt \\
&\quad + \sum_{k=1}^m \sum_{i=1}^n \int_0^1 \left(A^{ki}(x, t\theta)\, u_{\lambda x_i}^k(x, t\theta) \cdot \theta + A_{x_i}^{ki}(x, t\theta)\, u_\lambda^k(x, t\theta) \cdot \theta \right) dt \\
&= \sum_{k=1}^m \sum_{i=1}^n \int_0^1 \left(A^{ki}(x, t\theta) u_\lambda^k(x, t\theta) \cdot t\theta_{x_i} \right)_t \, dt \\
&\quad + \sum_{k=1}^m \sum_{i=1}^n \int_0^1 A^{ki}(x, t\theta)\, u_{\lambda x_i}^k(x, t\theta) \cdot \theta \, dt + \underbrace{\sum_{k=1}^m \int_0^1 L_{z^k}(D_x u(x, t\theta), u(x, t\theta), x)\, u_\lambda(x, t\theta) \cdot \theta \, dt}_{\text{By (7.15)}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\left(L(D_x u(x, t\theta), u(x, t\theta), x) \right)_t &= \sum_{k=1}^m \sum_{i=1}^n L_{p_i^k}(D_x u(x, t\theta), u(x, t\theta), x)\, u_{\lambda x_i}(x, t\theta) \cdot \theta \\
&\quad + \sum_{k=1}^m L_{z^k}(D_x u(x, t\theta), u(x, t\theta), x)\, u_\lambda(x, t\theta) \cdot \theta.
\end{aligned}$$

Combining the last two displays, we have

$$\begin{aligned}
\operatorname{div} b(x) &= \sum_{k=1}^m \sum_{i=1}^n A^{ki}(x, \theta) u_\lambda^k(x, \theta) \cdot \theta_{x_i} + L(D_x(u, \theta), u(x, \theta), x) - L(D_x u(x, 0), u(x, 0), x) \\
&= L_P(D_x(u, \theta), w, x) \cdot u_\lambda(x, \theta) D\theta + L(D_x(u, \theta), w, x) - L(Du, u, x). \qquad (7.17)
\end{aligned}$$

Step III. We combine (7.16) and (7.17), and apply Gauss-Green Theorem to obtain

$$\begin{aligned}
I[w] &\geq \int_U \left(\operatorname{div} b(x) + L(Du, u, x) \right) dx \\
&= \int_{\partial U} b(x) \cdot \nu \, dS + I[u] = I[u],
\end{aligned}$$

since $b \equiv 0$ on the boundary ∂U. Thus we complete the proof. $\square$

Remark. By the implicit function theorem, if

$$u_\lambda(x, 0) \neq 0$$

for all $x \in \overline{U}$, then we can write any w that is sufficiently close to u pointwise in this form.

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