A Derivation of Non-central Chi-square Density

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Non-central chi-square distribution. Let $(X_1, \cdots, X_p)$ be p independent Gaussian random variables with $X_i \sim N(\mu_i, 1)$. The random variable $V = \sum_{i=1}^p X_i^2$ is distributed according to the non-central chi-squared distribution, with degree of freedom p and noncentrality parameter $\lambda = \sum_{i=1}^p \mu_i^2$.

Main Theorem. (Density of non-central chi-square distributions). Suppose V is a non-central chi-square variable with degree of freedom $p > 0$ and non-centrality parameter $\lambda > 0$. Then the probability density function of V is

$$f_{\mathrm{NC}}(v; p, \lambda) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\lambda/2}(\lambda/2)^k}{k!} f_{\chi^2}(v; p + 2k), \tag{1}$$

where $f_{\chi^2}(\cdot; p + 2k)$ is the probability density function of $\chi^2(p + 2k)$:

$$f_{\chi^2}(v; p + 2k) = \frac{v^{p/2+k-1}}{2^{p/2+k}\Gamma(p/2 + k)} \mathrm{e}^{-v/2}.$$

I use the characteristic function method to derive the formula (1). First let's introduce some lemmas.

Lemma 1. Suppose $X \sim N(\mu, 1)$. Then the characteristic function of X^2 is

$$h(t; \mu) = \frac{\exp\left(\frac{\mathrm{i}\mu^2 t}{1-2\mathrm{i}t}\right)}{(1 - 2\mathrm{i}t)^{1/2}},\ t \in \mathbb{R}.$$

Proof. By definition, the characteristic function of X^2 is

$$\begin{aligned} h(t) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}tX^2}\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{\mathrm{i}tx^2 - \frac{(x-\mu)^2}{2}\right\} \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[(1-2\mathrm{i}t)^{1/2}x - \frac{\mu}{(1-2\mathrm{i}t)^{1/2}}\right]^2 + \frac{\mathrm{i}\mu^2 t}{1-2\mathrm{i}t}\right\} \mathrm{d}x \\ &= \exp\left(\frac{\mathrm{i}\mu^2 t}{1-2\mathrm{i}t}\right) \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[(1-2\mathrm{i}t)^{1/2}x - \frac{\mu}{(1-2\mathrm{i}t)^{1/2}}\right]^2\right\} \mathrm{d}x}_{\text{(a)}}, \end{aligned} \tag{2}$$

where the term (a) is $(1 - 2\mathrm{i}t)^{-1/2}$ as a Gaussian integral with complex shift. □

Lemma 2. Suppose V is a non-central chi-square variable with degree of freedom $p > 0$ and non-centrality parameter $\lambda > 0$. Then V can be represented as

$$V = Z_1^2 + Z_2^2 + \cdots + Z_p^2,\ Z_1 \sim N(\sqrt{\lambda}, 1),\ Z_2, \cdots, Z_p \sim N(0, 1), \tag{3}$$

where $Z_1, \cdots, Z_p$ are independent.

Proof. Let $X_i \sim N(\mu_i, 1), i = 1, \cdots, p$, with $\lambda = \mu_1^2 + \cdots + \mu_p^2 > 0$. Denote by $\mathbf{X}$ the random vector composed of $X_1, \cdots, X_p$. By definition,

$$V = X_1^2 + X_2^2 + \cdots + X_p^2 = \|\mathbf{X}\|_2^2, \ \mathbf{X} \sim N(\boldsymbol{\mu}, \mathbf{I}_p),$$

where $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)^\top \in \mathbb{R}^p$. Then we can expand $\lambda^{-1/2}\boldsymbol{\mu}$ to an orthogonal matrix $\mathbf{Q} = \begin{bmatrix} \lambda^{-1/2}\boldsymbol{\mu}^\top \\ * \end{bmatrix}$ of which the rows form an orthonormal basis on $\mathbb{R}^p$. Let $\mathbf{Z} = \mathbf{QX}$. Then

$$Z_1^2 + Z_2^2 + \cdots + Z_p^2 = \|\mathbf{Z}\|_2^2 = \|\mathbf{QX}\|_2^2 = \|\mathbf{X}\|_2^2 = V.$$

Moreover, $Z_1, Z_2, \cdots, Z_p$ are independent Gaussian variables characterized by (3). □

Lemma 3 (Convolution theorem). Let X, Y be independent random variables. Then the characteristic function of $X + Y$ is the pointwise product of the characteristic functions of X and Y.

Now we are prepared to prove the Main Theorem in the beginning.

Proof of Main Theorem. We use the representation of V given by Lemma 2. Then applying Lemmas 1 and 3 yields the characteristic function of V :

$$\varphi_V(t) = h(t; \sqrt{\lambda}) \prod_{i=2}^{p} h(t; 0) = \frac{\exp\left(\frac{\mathrm{i}\lambda t}{1-2\mathrm{i}t}\right)}{(1-2\mathrm{i}t)^{p/2}}, \ t \in \mathbb{R}.$$

We can expand the numerator as follows:

$$\exp\left(\frac{\mathrm{i}\lambda t}{1-2\mathrm{i}t}\right) = \mathrm{e}^{-\lambda/2} \exp\left(\frac{\lambda/2}{1-2\mathrm{i}t}\right) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\lambda/2}}{k!} \left(\frac{\lambda/2}{1-2\mathrm{i}t}\right)^k.$$

Then

$$\varphi_V(t) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\lambda/2}(\lambda/2)^k}{k!} \underbrace{\frac{1}{(1-2\mathrm{i}t)^{p/2+k}}}_{\text{CF of } \chi^2(p+2k)}. \tag{4}$$

Applying Fourier transform on both sides of (4) yields the result of (1). □

Remark. This theorem proposes another method of generating non-central chi-square variables. Fix $p, \lambda > 0$.
- Generate $k \sim \text{Poisson}(\lambda/2)$.
- Generate i.i.d. $X_1, \cdots, X_{p+2k} \sim N(0, 1)$, and set $V = X_1^2 + \cdots + X_{p+2k}^2$.

Then V is a non-central chi-square variable with degree of freedom p and non-centrality parameter λ.

References

[Anderson 2003] Theodore W. Anderson. (2003). *An Introduction to Multivariate Statistical Analysis.* John Wiley & Sons Inc; 3rd Edition.

Appendix: Gaussian integral with complex arguments

In this section, I provide a detailed calculation of term (a) in (2).

Step I. Calculate the centered Gaussian integral

We consider the following improper integral:

$$G_\gamma := \int_{-\infty}^{\infty} \mathrm{e}^{-\gamma x^2}\mathrm{d}x, \quad \gamma \in \mathbb{C},\ \mathrm{Re}(\gamma) > 0.$$

The indefinite integral of function $\mathrm{e}^{-\gamma x^2}$ is intractable, but we can transform the integrand by changing one variable in a squared form:

$$G_\gamma^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\gamma(x^2+y^2)}\mathrm{d}x\mathrm{d}y. \tag{A.1}$$

Then we can compute (A.1) in the $\mathbb{R}^2$ plane by converting from Cartesian to polar coordinates:

$$\begin{aligned} G_\gamma^2 &= \int_{\mathbb{R}^2} \mathrm{e}^{-\gamma(x^2+y^2)}\mathrm{d}x\mathrm{d}y = \int_0^{2\pi}\int_0^{\infty} r\mathrm{e}^{-\gamma r^2}\mathrm{d}r\mathrm{d}\theta = \pi\int_0^{\infty}\mathrm{e}^{-\gamma t}\mathrm{d}t \qquad \text{(By changing variable } t = r^2\text{)} \\ &= -\frac{\pi \mathrm{e}^{-\gamma t}}{\gamma}\bigg|_0^{\infty} = \frac{\pi}{\gamma}. \end{aligned} \tag{A.2}$$

where the last equality holds because $\mathrm{Re}(\gamma) > 0$.

Denote $\gamma = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then the equation (A.2) has two conjugate solutions

$$G_\gamma = \sqrt{\frac{\pi}{r}}\,\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}} \quad \text{or} \quad G_\gamma = -\sqrt{\frac{\pi}{r}}\,\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}.$$

Note that $G_\gamma = 1$ when $\theta = 0 \Leftrightarrow \gamma \in \mathbb{R}$, we have $G_\gamma > 0$. Following the continuity of G_γ, the first solution is correct. We denote $\gamma^{1/2} = \sqrt{r}\mathrm{e}^{\mathrm{i}\theta/2}$ by the square root of π of which the real part is positive:

$$G_\gamma = \frac{\sqrt{\pi}}{\gamma^{1/2}}.$$

Step II. Real shift

For any $\alpha \in \mathbb{R}$, by changing the variable, we have

$$G_{\alpha,\gamma} := \int_{-\infty}^{\infty} \mathrm{e}^{-\gamma(x+\alpha)^2}\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{-\gamma x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{\gamma^{1/2}}, \quad \gamma \in \mathbb{C},\ \mathrm{Re}(\gamma) > 0. \tag{A.3}$$

Step III. Complex shift

Now we calculate the Gaussian integral with a complex shift $\alpha + \mathrm{i}\beta$:

$$G_{\alpha,\beta,\gamma} := \int_{-\infty}^{\infty} \mathrm{e}^{-\gamma(x+\alpha+\mathrm{i}\beta)^2}\mathrm{d}x. \quad \gamma \in \mathbb{C},\ \mathrm{Re}(\gamma) > 0,\ \alpha \in \mathbb{R},\ \beta \in \mathbb{R}.$$

Without loss of generality, we suppose $\beta > 0$. We fix a number $T > 0$, and construct a contour C composed of line segments $-T \to T \to T + \mathrm{i}\beta \to -T + \mathrm{i}\beta \to -T$, as is shown in Figure 1.

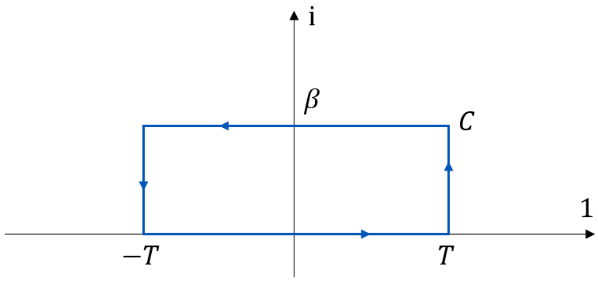

Figure 1: Integral path

By Cauchy's integral theorem (See [Howie 2003]), the integral of function $f(z) := \mathrm{e}^{-\gamma(z+\alpha)^2}$, which is holomorphic in the complex plane $\mathbb{C}$, is zero along any simply closed contour. Then

$$\begin{aligned} 0 &= \oint_C \mathrm{e}^{-\gamma(z+\alpha)^2}\mathrm{d}z \\ &= \underbrace{\int_{-T}^{T} \mathrm{e}^{-\gamma(x+\alpha)^2}\mathrm{d}x}_{(i)} + \underbrace{\int_{0}^{\beta} \mathrm{e}^{-\gamma(T+\mathrm{i}y+\alpha)^2}\mathrm{d}y}_{\text{(ii)}} + \underbrace{\int_{T}^{-T} \mathrm{e}^{-\gamma(x+\mathrm{i}\beta+\alpha)^2}\mathrm{d}x}_{\text{(iii)}} + \underbrace{\int_{\beta}^{0} \mathrm{e}^{-\gamma(-T+\mathrm{i}y+\alpha)^2}\mathrm{d}y}_{\text{(iv)}}. \end{aligned} \tag{A.4}$$

Now let $T \to \infty$, then $\exp\left\{-\gamma(\pm T + \mathrm{i}y + \alpha)^2\right\} \to 0$. The integrals (ii) and (iv), defined on a bounded interval $[0, \beta]$, converge to zero, and (A.4) reduces to

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\gamma(x+\mathrm{i}\beta+\alpha)^2}\mathrm{d}x = \int_{-\infty}^{\infty} \mathrm{e}^{-\gamma(x+\alpha)^2}\mathrm{d}x.$$

Combining with (A.3), we proved that for any $\gamma, \mu \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$, we have

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\gamma(x-\mu)^2}\mathrm{d}x = \frac{\sqrt{\pi}}{\gamma^{1/2}}. \tag{A.5}$$

Step IV. Plugging-in

Using formula (A.5), we can immediately calculate the term (a) in (2):

$$\begin{aligned} \text{(a)} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1-2\mathrm{i}t}{2}\left(x - \frac{\mu}{1-2\mathrm{i}t}\right)^2\right\}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{(1-2\mathrm{i}t)^{1/2}/\sqrt{2}} = \frac{1}{(1-2\mathrm{i}t)^{1/2}}. \end{aligned}$$

Then we complete the entire proof of Lemma 1.

References

[Howie 2003] John M. Howie (2003). Cauchy's Theorem. In *Complex Analysis*. Springer Undergraduate Mathematics Series. Springer, London.