Fourier Analysis and Distribution Theory

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1 Preliminaries

1.1 Convolution

In this section we study the convolution operation on $\mathbb{R}^n$. If a function f is defined on $U \subset \mathbb{R}^n$, we can replace it by its natural zero extension $f : \mathbb{R}^n \to \mathbb{R}$ which assigns $f(x) = 0$ for $x \notin U$.

Definition 1.1 (Convolution)**.** Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions. Define the bad set as

$$E(f,g) := \left\{ x \in \mathbb{R}^n : \int_{\mathbb{R}^n} |f(x-y)g(y)|\, dy = \infty \right\}.$$

The *convolution* of f and g is the function $f * g : \mathbb{R}^n \to \mathbb{R}$ defined by

$$(f*g)(x) = \begin{cases} \int_{\mathbb{R}^n} f(x-y)g(y)\, dy, & x \notin E(f,g), \\ 0, & x \in E(f,g). \end{cases}$$

Remark. Define $F : \mathbb{R}^{2n} \to \mathbb{R}, (x,y) \mapsto f(x)$ and $G : \mathbb{R}^{2n} \to \mathbb{R}, (x,y) \mapsto g(y)$. Then both F and G are measurable functions on $\mathbb{R}^{2n}$, as well as their product $F \cdot G : (x,y) \mapsto f(x)g(y)$. Given linear transformation $T(x,y) = (x-y, y)$, the composition $H = (F \cdot G) \circ T : (x,y) \mapsto f(x-y)g(y)$ is measurable. By Tonelli's theorem, the function $x \mapsto \int_{\mathbb{R}^n} |H(x,y)|\, dy$ is measurable, and $E(f,g)$ is a Lebesgue measurable set.

Clearly, the convolution operation is both commutative and associative, i.e. $f * g = g * f$, and $(f*g)*h = f*(g*h)$. Furthermore, the distributivity of convolution with respect to functional addition immediately follows, i.e. $f*(g+h) = f*g + f*h$.

Proposition 1.2 (Properties of convolution)**.** *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable functions.*

(i) If $f, g \in L^1(\mathbb{R}^n)$, then the bad set $E(f,g)$ is of measure zero. Moreover, $f*g \in L^1(\mathbb{R}^n)$, and

$$\int_{\mathbb{R}^m} (f*g)\, dm = \int_{\mathbb{R}^n} f\, dm \int_{\mathbb{R}^n} g\, dm. \tag{1.1}$$

(ii) If $f \in C_u(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f*g \in C_u(\mathbb{R}^n)$.

(iii) If $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$, then $f*g \in L^p(\mathbb{R}^n)$, and

$$\|f*g\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^1}.$$

Proof. (i) Define the measurable function $H(x,y) \mapsto f(x-y)g(y)$ on $\mathbb{R}^{2n}$. By Tonelli's theorem,

$$\int_{\mathbb{R}^{2n}} |H|\, dm = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|\, |g(y)|\, dx \right) dy = \|f\|_{L^1}\|g\|_{L^1}.$$

Hence $H : \mathbb{R}^{2n} \to \mathbb{R}$ is integrable. By Fubini's theorem, for a.e. $x \in \mathbb{R}^n$, $y \mapsto H(x,y)$ is integrable, hence $m(E(f,g)) = 0$. Furthermore, the function $f*g : x \mapsto \int_{\mathbb{R}^n} H(x,y)\, dy$ is also integrable, that is, $f*g \in L^1(\mathbb{R}^n)$. The equation (1.1) follows from Fubini's theorem.

(ii) Given $\epsilon > 0$. By uniform continuity of f, there exists $\eta > 0$ such that $|f(x) - f(x')| < \epsilon/\|g\|_{L^1}$ for all $|x - x'| < \eta$, . As a result, for all $x, x' \in \mathbb{R}^n$ such that $|x - x'| < \eta$, we have

$$|(f*g)(x) - (f*g)(x')| \le \int_{\mathbb{R}^n} |f(x-y) - f(x'-y)|\, |g(y)|\, dy < \epsilon.$$

(iii) is a special case of the following proposition. ☐

Proposition 1.3 (Young's convolution inequality)**.** *Given $r \in [1,\infty]$ and Hölder r-conjugates $p,q \in [1,\infty]$, i.e. $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then the bad set $E(f,g)$ is of measure zero, and we have*

$$\|f*g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}.$$

Remark. Note that $r=\frac{pq}{p+q-pq}\ge 1 \;\Leftrightarrow\; \frac{pq}{p+q}\ge\frac{1}{2} \;\Leftrightarrow\; p\ge\frac{q}{2q-1} \;\Leftrightarrow\; q\ge\frac{p}{2p-1}$,
and $r<\infty \;\Leftrightarrow\; p+q>pq \;\Leftrightarrow\; p<\frac{q}{q-1} \;\Leftrightarrow\; q<\frac{p}{p-1}$.

Proof. We first bound $f*g$. By applying generalized Hölder's inequality on $\frac{1}{r}+\frac{r-p}{pr}+\frac{r-q}{qr}=1$, we have

$$\begin{aligned}
|(f*g)(x)| &\le \int_{\mathbb{R}^n}|f(x-y)|\,|g(y)|\;dy\\
&= \int_{\mathbb{R}^n}\left(|f(x-y)|^p|g(y)|^q\right)^{1/r}|f(x-y)|^{\frac{r-p}{r}}|g(y)|^{\frac{r-q}{r}}\;dy\\
&\le \left(\int_{\mathbb{R}^n}|f(x-y)|^p|g(y)|^q\,dy\right)^{1/r}\left(\int_{\mathbb{R}^n}|f(x-y)|^p\,dy\right)^{\frac{r-p}{pr}}\left(\int_{\mathbb{R}^n}|g(y)|^q\,dy\right)^{\frac{r-q}{qr}}\\
&= \left(\int_{\mathbb{R}^n}|f(x-y)|^p|g(y)|^q\,dy\right)^{1/r}\|f\|_{L^p}^{\frac{r-p}{r}}\,\|g\|_{L^q}^{\frac{r-q}{r}}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x-y)|\,|g(y)|\;dy\right)^r dx &\le \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(x-y)|^p|g(y)|^q\,dy\,dx\right)\|f\|_{L^p}^{r-p}\,\|g\|_{L^q}^{r-q}\\
&\le \|f\|_{L^p}^{r-p}\,\|g\|_{L^q}^{r-q}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|f(x-y)|^p\,dx\right)|g(y)|^q\,dy = \|f\|_{L^p}^r\,\|g\|_{L^q}^r,
\end{aligned}$$

where we use Fubini's theorem in the second inequality. From the last display, we have $m(E(f,g))=0$, and $\|f*g\|_{L^r}\le\|f\|_{L^p}\|g\|_{L^q}$. □

Remark. If $f\in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, and $g\in L^q(\mathbb{R}^n)$ is compactly supported, then $f*g\in L^r_{\mathrm{loc}}(\mathbb{R}^n)$.

Proposition 1.4 (Convolution of compactly supported functions)**.** *Let $f,g:\mathbb{R}^n\to\mathbb{R}$.*

(i) If $f,g\in L^1(\mathbb{R}^n)$, then $\operatorname{supp}(f*g)\subset\overline{\operatorname{supp}f+\operatorname{supp}g}:=\overline{\{x+y:x\in\operatorname{supp}f,y\in\operatorname{supp}g\}}$. Furthermore, if both f and g are compactly supported on $\mathbb{R}$, then $f*g$ is also compactly supported. In this case, $\operatorname{supp}(f*g)\subset\operatorname{supp}f+\operatorname{supp}g$.

(ii) Let $1\le p\le\infty$, and let $k\in\mathbb{N}_0$. If $f\in C_c^k(\mathbb{R}^n)$ and $g\in L^p(\mathbb{R}^n)$, then $f*g\in C_u^k(\mathbb{R}^n)$. Furthermore, differentiation commutes with convolution, i.e.,

$$\partial^\alpha(f*g)=\partial^\alpha f*g,\qquad\forall|\alpha|\le k,$$

(iii) Let $1\le p\le\infty$. If $f\in C_c^\infty(\mathbb{R}^n)$ and $g\in L^p(\mathbb{R}^n)$, then $f*g\in C_u^\infty(\mathbb{R}^n)$. Similarly, differentiation commutes with convolution, i.e., $\partial^\alpha(f*g)=\partial^\alpha f*g$ for multi-indices α.

Remark. Here is a slight modification of assertions (ii) and (iii):

(ii') Let $1\le p\le\infty$, and let $k\in\mathbb{N}_0$. If $f\in C_c^k(\mathbb{R}^n)$ and $g\in L^p(\mathbb{R}^n)$, then $f*g\in C_u^k(\mathbb{R}^n)$. Furthermore, differentiation commutes with convolution, i.e.,

$$\partial^\alpha(f*g)=\partial^\alpha f*g,\qquad\forall|\alpha|\le k,$$

(iii') Let $1\le p\le\infty$. If $f\in C_c^\infty(\mathbb{R}^n)$ and $g\in L^p(\mathbb{R}^n)$, then $f*g\in C_u^\infty(\mathbb{R}^n)$. Similarly, differentiation commutes with convolution, i.e., $\partial^\alpha(f*g)=\partial^\alpha f*g$ for multi-indices α.

Proof. (i) Let $f, g \in L^1(\mathbb{R}^n)$, and take any $x \in \mathbb{R}^n$. Then

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy = \int_{(x-\operatorname{supp} f)\cap \operatorname{supp} g} f(x-y)g(y)\,dy.$$

For $x \notin \operatorname{supp} f + \operatorname{supp} g$, we have $(x - \operatorname{supp} f) \cap \operatorname{supp} g = \emptyset$, which implies $(f * g)(x) = 0$. Hence

$$(f * g)(x) \neq 0 \ \Rightarrow x \in \operatorname{supp} f + \operatorname{supp} g \ \Rightarrow \ \operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}.$$

If $f, g \in C_c(\mathbb{R}^n)$, then $\operatorname{supp} f$ and $\operatorname{supp} g$ are compact in $\mathbb{R}^n$. Define $\phi(x, y) = x + y$, which is a continuous map on $\mathbb{R}^n \times \mathbb{R}^n$. Then $\operatorname{supp} f + \operatorname{supp} g = \phi(\operatorname{supp} f \times \operatorname{supp} g)$ is also compact. Consequently, $\operatorname{supp} f + \operatorname{supp} g$ is closed, and its closed subset $\operatorname{supp}(f * g)$ is also compact. which implies $f * g \in C_c(\mathbb{R}^n)$.

(ii) *Step I:* We first show the case $k = 0$. Let $q = p/(p-1)$. Note that f is continuous and compact supported, then $m(\operatorname{supp} f) < \infty$, f is uniformly continuous, and $\|f\|_\infty = \max_{x\in \operatorname{supp} f} |f(x)| < \infty$. By Hölder's inequality, for all $x \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} |f(x-y)|\,|g(y)|\,dy \leq \|f\|_{L^q}\|g\|_{L^p} \leq m\big(\operatorname{supp} f\big)^{1/q}\|f\|_\infty \|g\|_{L^p} < \infty.$$

Then $f * g$ is well-defined on $\mathbb{R}^n$. To show uniform continuity of $f * g$, we fix $\epsilon > 0$ and let η be such that $|x - x'| < \eta$ implies $|f(x) - f(x')| < \epsilon$. Then

$$\begin{aligned} |(f * g)(x) - (f * g)(x')| &= \left|\int_{\mathbb{R}^n} [f(x-y) - f(x'-y)]\, g(y)\,dy\right| \\ &\leq 2m\big(\operatorname{supp} f\big)^{1/q} \|g\|_{L^p}\,\epsilon. \end{aligned}$$

Step II: We prove the case $k = 1$. It suffices to show the interchangeability of derivative and integral. Given any quantity $h > 0$, we have

$$\frac{(f * g)(x + he_i) - (f * g)(x)}{h} = \int_{\mathbb{R}^n} \frac{f(x + he_i - y) - f(x - y)}{h} g(y)\,dy. \tag{1.2}$$

Since $f \in C_c^1(\mathbb{R}^n)$, by Lagrange's mean value theorem, there exists $\xi \in [0, 1]$ such that

$$\left|\frac{f(x + he_i - y) - f(x - y)}{h}\right| = |\partial_{x_i} f(x + \xi h e_i - y)|\,, \tag{1.3}$$

Note that $\partial_{x_i} f$ is also continuous and compactly supported on $\mathbb{R}^n$, the RHS of (1.3) is bounded by $\|\partial_{x_i} f\|_\infty$, and the integrand in (1.2) is dominated by an integrable function $\|\partial_{x_i} f\|_\infty g$. Using Lebesgue's dominate convergence theorem, we have

$$\lim_{h\to 0} \int_{\mathbb{R}^n} \frac{f(x + he_i - y) - f(x - y)}{h} g(y)\,dy = \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x - y)g(y)\,dy.$$

Therefore $\partial_{x_i}(f * g) = \partial_{x_i} f * g$. Since $\partial_{x_i} f \in C_c(\mathbb{R}^n)$, we have $\partial_{x_i}(f * g) \in C_u(\mathbb{R}^n)$, and $f * g \in C_u^1(\mathbb{R}^n)$.

Step III: Use induction. Suppose our conclusion holds for $C_c^{k-1}(\mathbb{R}^n)$. For each $f \in C_c^k(\mathbb{R}^n) \subset C_c^{k-1}(\mathbb{R}^n)$, $\partial^{k-1} f \subset C_c^1(\mathbb{R}^n)$. By Step II, for any $|\alpha| = k - 1$,

$$\partial^{\alpha+e_i}(f * g) = \partial_{x_i}(\partial^\alpha(f * g)) = \partial_{x_i}(\partial^\alpha f * g) = (\partial^{\alpha+e_i} f) * g,$$

which is uniformly continuous on $\mathbb{R}^n$. Hence $f * g \in C_u^k(\mathbb{R}^n)$.

(iii) Note that $C_c^\infty(\mathbb{R}^n) = \bigcap_{k=0}^\infty C_c^k(\mathbb{R}^n)$, we have $\partial^\alpha(f * g) = \partial^\alpha f * g$ for all $\alpha \in \mathbb{N}_0^n$. Following Step II, $\partial^\alpha f \in C_c(\mathbb{R}^n)$ implies $\partial^\alpha(f * g) \in C_u(\mathbb{R}^n)$ for all $\alpha \in \mathbb{N}_0^n$. Hence $f * g \in \bigcap_{k=0}^\infty C_u^k(\mathbb{R}^n) = C_u^\infty(\mathbb{R}^n)$. □

Translation operators. Let X be a vector space, let Y^X be the set of functions $f : X \to Y$, and let s be a vector in X. The *translation operator* $\tau_s : Y^X \to Y^X$ is defined as

$$(\tau_s f)(x) = f(x - s), \ \forall f \in Y^X.$$

The following proposition gives a description of the continuity of $(\tau_s)_{s \in X}$ in C_c and L^p spaces.

Proposition 1.5. *Let $1 \leq p < \infty$.*
(i) For any $f \in C_c(\mathbb{R}^n)$, $\tau_s f \to f$ uniformly and in L^p-norm as $s \to 0$.
(ii) For any $f \in L^p(\mathbb{R}^n)$, $\tau_s f \to f$ in L^p-norm as $s \to 0$.

Proof. Let $f \in C_c(\mathbb{R}^n)$, and let $B_1 = \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the compact unit ball in $\mathbb{R}^n$. The collection of functions $\{\tau_s f : |s| \leq 1\}$ has a common support

$$K = \bigcup_{|s| \leq 1} \operatorname{supp}(\tau_s f) = \operatorname{supp} f + B_1 = \{x + y : x \in \operatorname{supp} f, y \in B_1\}.$$

Since the addition operation is continuous, K is also a compact subset of $\mathbb{R}^n$.

By uniform continuity of f, given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $|x - y| < \delta$. Hence $\tau_s f \to f$ uniformly as $s \to 0$. Moreover, for any s with $|s| < |\min(\delta, 1)|$, we have

$$\|\tau_s f - f\|_{L^p}^p = \int_K |f(x - s) - f(x)|^p dx \leq \mu(K)\, \epsilon^p.$$

Since $\mu(K) < \infty$, and ϵ is arbitrary, we conclude that $\|\tau_s f - f\|_{L^p} \to 0$ as $s \to 0$.

Now we assume $f \in L^p(\mathbb{R}^n)$, and fix $\epsilon > 0$. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, there exists $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_\infty < \epsilon/3$. Choose δ such that $\|\tau_s g - g\|_{L^p} < \epsilon/3$ for all $|s| < \delta$. Then for all $|s| < \delta$,

$$\|\tau_s f - f\|_{L^p} \leq \|\tau_s f - \tau_s g\|_{L^p} + \|\tau_s g - g\|_{L^p} + \|g - f\|_{L^p} = 2\|f - g\| + \|\tau_s g - g\|_{L^p} < \epsilon.$$

Therefore, $\lim_{s \to 0} \|\tau_s f - f\|_{L^p} = 0$ for all $f \in L^p(\mathbb{R}^n)$. ☐

Proposition 1.6 (Mollification). *Let $\phi \in L^1(\mathbb{R}^n)$, with $\int_{\mathbb{R}^n} \phi\, dx = a$. Given $t > 0$, define*

$$\phi_t(x) = \frac{1}{t^n} \phi\left(\frac{x}{t}\right). \tag{1.4}$$

(i) If $f \in L^p(\mathbb{R}^n)$, $f * \phi_t \to af$ in $L^p(\mathbb{R}^n)$ as $t \to 0$.
(ii) If f is bounded and uniformly continuous, $f * \phi_t \to af$ uniformly as $t \to 0$.

Proof. Using the decomposition $\phi = \phi^+ - \phi^-$, we may assume $\phi \geq 0$ on $\mathbb{R}^n$. We further assume $a = 1$ by replacing ϕ by ϕ/a if necessary. Then

$$(f * \phi_t)(x) - f(x) = \int_{|y| \leq t} (f(x - y) - f(x))\phi_t(y)\, dy = \int_{|y| \leq t} (\tau_y f - f)(x)\phi_t(y)\, dy.$$

By Jensen's inequality and Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} |(f * \phi_t)(x) - f(x)|^p\, dx &= \int_{\mathbb{R}^n} \left| \int_{|y| \leq t} (\tau_y f - f)(x)\phi_t(y)\, dy \right|^p dx \\ &\leq \int_{\mathbb{R}^n} \int_{|y| \leq t} |\tau_y f(x) - f(x)|^p\, \phi_t(y)\, dy\, dx \leq \sup_{|y| < t} \|\tau_y f - f\|_{L^p}. \end{aligned}$$

By continuity of the translation operator, the first result follows. For the second result, use the same estimate for $f * \phi_t - f$ and the uniform continuity of f. ☐

When we establish the density arguments of C_c^∞ functions, the above result is very useful.

Proposition 1.7. *For $1 \le p < \infty$, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$.*

Proof. By the first assertion in Proposition 1.6, $C_c^\infty(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R})$ in $\|\cdot\|_1$ norm. Since $C_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, the result follows. □

Proposition 1.8. *For $1 \le p < \infty$, $C_c^\infty(\mathbb{R}^n)$ is dense in $C_0(\mathbb{R}^n)$.*

Proof. By the second assertion in Proposition 1.6, $C_c^\infty(\mathbb{R}^n)$ is dense in $C_c(\mathbb{R})$ in $\|\cdot\|_\infty$ norm. Since $C_0(\mathbb{R}^n)$ is the closure of $C_c(\mathbb{R}^n)$ in $\|\cdot\|_\infty$ norm, the result follows. □

Aside from the convergence in L^p-norm discussed in Proposition 1.6, we are also interested in the pointwise convergence property of mollification $f * \phi_\epsilon$.

Proposition 1.9 (Mollification)**.** *Assume $\phi \in L^1(\mathbb{R}^n)$ satisfies $|\phi(x)| \le C(1+|x|)^{-n-\gamma}$ for some $C, \gamma > 0$, and $\int_{\mathbb{R}^n} \phi\, dx = a$. Define ϕ_ϵ as in (1.4). Let $1 \le p \le \infty$. If $f \in L^p(\mathbb{R}^n)$, then $(f * \phi_\epsilon)(x) \to af(x)$ as $\epsilon \to 0$ for every Lebesgue point x of f.*

Proof. If x is a Lebesgue point of f, we have

$$\lim_{r\to 0^+} \frac{1}{r^n} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0.$$

For any $\epsilon > 0$, we choose $\delta > 0$ such that $\int_{B(x,r)} |f(y) - f(x)|\, dy < r^n \epsilon$ for all $r \le \delta$, and set

$$I_1 = \int_{|y|<\delta} |f(x-y) - f(x)|\, |\phi_t(y)|\, dy, \quad I_2 = \int_{|y|\ge\delta} |f(x-y) - f(x)|\, |\phi_t(y)|\, dy.$$

We claim that I_1 is bounded by $A\epsilon$, where A is independent of t, and $I_2 \to 0$ as $t \to 0$. Since

$$|(f * \phi_t)(x) - af(x)| \le I_1 + I_2,$$

we will have

$$\limsup_{t\to 0^+} |(f * \phi_t)(x) - af(x)| \le A\epsilon,$$

Since $\epsilon > 0$ is arbitrary, the proof will be completed.

To estimate I_1, let N be the integer such that $2^N \le \delta/t < 2^{N+1}$, if $\delta/t \ge 1$, and $N = 0$ if $\delta/t < 1$. We view the ball $|y| < \delta$ as the union of the annuli $2^{-k}\delta \le |y| < 2^{1-k}\delta$, $1 \le k \le N$ and the ball $|y| < 2^{-N}\delta$. On the k^{th} annulus we use the estimate

$$|\phi_t(y)| = \frac{1}{t^n} \left|\phi\left(\frac{y}{t}\right)\right| \le Ct^{-n} \left|\frac{y}{t}\right|^{-n-\gamma} \le Ct^{-n} \left(\frac{2^{-k}\delta}{t}\right)^{-n-\gamma}$$

and in the ball $|y| < 2^{-N}\delta$, we use the estimate $|\phi_t(y)| \le Ct^{-n}$. Thus

$$\begin{aligned} I_1 &\le \sum_{k=1}^N Ct^{-n} \left(\frac{2^{-k}\delta}{t}\right)^{-n-\gamma} \int_{2^{-k}\delta \le |y| < 2^{1-k}\delta} |f(x-y) - f(x)|\, dy + Ct^{-n} \int_{|y|<2^{-N}\delta} |f(x-y) - f(x)|\, dy \\ &\le C\epsilon \sum_{k=1}^N (2^{1-k}\delta)^n t^{-n} \left(\frac{2^{-k}\delta}{t}\right)^{-n-\gamma} + C\epsilon (2^{-N}\delta)^n t^{-n} = 2^n C\epsilon \left(\frac{\delta}{t}\right)^{-\gamma} \sum_{k=1}^N 2^{k\gamma} + C\epsilon \left(\frac{2^{-N}\delta}{t}\right)^n \\ &= 2^n C\epsilon \left(\frac{\delta}{t}\right)^{-\gamma} \frac{2^{(N+1)\gamma} - 2^\gamma}{2^\gamma - 1} + C\epsilon \left(\frac{2^{-N}\delta}{t}\right)^n \le \underbrace{2^n C\left(\frac{2^\gamma}{2^\gamma - 1} + 1\right)}_{=:A} \epsilon. \end{aligned}$$

As for I_2, if q is the conjugate exponent to p and χ is the characteristic function of the set $\{y \in \mathbb{R}^n : |y| \geq \delta\}$,

$$I_2 \leq \int_{|y|\geq\delta} (|f(y-x)| - |f(x)|) |\phi_t(y)| \, dy \leq \|f\|_{L^p} \|\chi\phi_t\|_{L^q} + |f(x)| \|\chi\phi_t\|_{L^1}.$$

If $q = \infty$,

$$\|\chi\phi_t\|_{L^\infty} \leq Ct^{-n} \left(1 + \frac{\delta}{t}\right)^{-n-\gamma} = \frac{Ct^\delta}{(t+\delta)^{n+\gamma}} \leq \frac{Ct^\delta}{\delta^{n+\gamma}},$$

which converges to 0 as $t \to 0$. If $1 \leq q < \infty$, we switch to the sphere coordinates:

$$\begin{aligned}
\|\chi\phi_t\|_{L^q} &= \int_{|y|\geq\delta} t^{-nq} \left|\phi\left(\frac{y}{t}\right)\right|^q dy = \int_{|z|\geq\delta/t} t^{n(1-q)} |\phi(z)|^q \, dz \\
&\leq C_n t^{n(1-q)} \int_{\delta/t}^\infty r^{n-1} C(1+r)^{-(n+\gamma)q} \, dr \\
&\leq C_n C t^{n(1-q)} \int_{\delta/t}^\infty r^{n-1-(n+\gamma)q} \, dr \\
&= C_n C t^{n(1-q)} \frac{(\delta/t)^{n-(n+\gamma)q}}{(n+\gamma)q - n} = \frac{C_n C \delta^{n-(n+\gamma)q} t^{\gamma q}}{(n+\gamma)q - n},
\end{aligned}$$

which also converges to 0 as $t \to 0$. Therefore $I_2 \to 0$ as $t \to 0$, and we are done. □

Finally we see an application of the mollification.

Proposition 1.10 (C^∞-Urysohn lemma)**.** *Let $U \subset \mathbb{R}^n$ be an open set, and let $K \subset U$ be a compact set. There exists $f \in C_c^\infty(U)$ such that $0 \leq f \leq 1$, and $f = 1$ on K.*

Proof. Since K is compact and U is open, we take $0 < \epsilon < d(K, U^c)$. Define

$$V = \left\{x \in U : d(x, K) \leq \frac{\epsilon}{3}\right\}, \quad \text{and} \quad W = \left\{x \in U : d(x, K) < \frac{2\epsilon}{3}\right\}.$$

Then V is a compact set, W is an open set, and $K \subset V^\circ \subset V \subset W \subset \overline{W} \subset U$. By Urysohn's lemma, there exists $g \in C_c(W)$ such that $0 \leq g \leq 1$ and $g = 1$ on V. Now we choose $\phi \in C_c^\infty(\mathbb{R}^n)$ such that ϕ is supported on the closed ball $\overline{B(0, \frac{\epsilon}{3})}$ and $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$. Then $f = g * \phi$ is the desired function. □

1.2 The Schwartz Space

Definition 1.11 (Schwartz space)**.** The *Schwartz space* consists of all C^∞-functions, which, together with their derivatives, vanishes at infinity faster than any power of $|x|$. More precisely, for any $f \in C^\infty(\mathbb{R}^n)$, any nonnegative integer N and any multi-index $\alpha \in \mathbb{N}_0^n$, define the norm

$$\|f\|_{(N,\alpha)} = \sup_{x\in\mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|.$$

The Schwartz space is

$$\mathcal{S}(\mathbb{R}^n) = \left\{f \in C^\infty(\mathbb{R}^n) : \|f\|_{(N,\alpha)} < \infty \; for \; all \; N \in \mathbb{N}_0, \; \alpha \in \mathbb{N}_0^n\right\}.$$

Remark. For any $\phi \in C_c^\infty(\mathbb{R}^n)$, all its derivatives are also C_c^∞, and

$$\|\phi\|_{(N,\alpha)} \leq \sup_{x\in \operatorname{supp}\phi} (1 + |x|)^N \|\partial^\alpha \phi\|_\infty < \infty.$$

Therefore, we have $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.

Proposition 1.12. *The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space under the topology induced by norms $\|\cdot\|_{(N,\alpha)}$.*

Proof. It suffices to show the completeness of $\mathcal{S}(\mathbb{R}^n)$. Let (f_k) be a Cauchy sequence in $\mathcal{S}(\mathbb{R}^n)$, which implies that $\|f_k - f_m\|_{(N,\alpha)} \to 0$ as $k, m \to \infty$ for all $N \in \mathbb{N}_0$ and all multi-indices $\alpha \in \mathbb{N}_0^n$. In particular, for each α, the sequence $(\partial^\alpha f_k)$ converges uniformly to a function g_α. We denote by $e_j = (0, \cdots, \underset{j\text{-th}}{1}, 0, \cdots, 0)$. Then

$$f_k(x + he_j) - f_k(x) = \int_0^h \frac{\partial f_k}{\partial x_j}(x + te_j)\, dt.$$

Letting $k \to \infty$ and apply dominated convergence theorem, we obtain $g_0(x+he_j) - g_0(x) = \int_0^h g_{e_j}(x+te_j)\, dt$, which implies that $\partial_{x_j} g_0 = g_{e_j}$ by the fundamental theorem of calculus. An inductive argument on $|\alpha|$ implies $D^\alpha g_0 = g_\alpha$. Then $\|f_k - g_0\|_{(N,\alpha)} \to 0$ for all $N \in \mathbb{N}_0$ and all $\alpha \in \mathbb{N}_0^n$. □

Proposition 1.13 (Characterization of Schwartz space)**.** *Let $f \in C^\infty(\mathbb{R}^n)$. The following are equivalent:*

(i) $f \in \mathcal{S}(\mathbb{R}^n)$;

(ii) For all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, the function $x^\beta \partial^\alpha f$ is bounded;

(iii) For all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, the function $\partial^\alpha(x^\beta f)$ is bounded.

Proof. To show (i) ⇒ (ii), note that $|x|^\beta \leq (1 + |x|)^N$ for $|\beta| \leq N$. On the other hand, if (ii) holds, we fix an order $N \in \mathbb{N}$ and a multi-index $\alpha \in \mathbb{N}_0^n$, and take

$$\delta_N = \min\left\{\sum_{j=1}^n |x_j|^N : |x|^2 = \sum_{j=1}^n |x_j|^2 = 1\right\} > 0.$$

By homogeneity, we have $\sum_{j=1}^n |x_j|^N \geq \delta_N |x|^N$ for all $x \in \mathbb{R}^n$, and

$$(1 + |x|)^N \leq 2^N \left(1 + |x|^N\right) \leq 2^N \left(1 + \frac{1}{\delta_N} \sum_{j=1}^n |x_j|^N\right) \leq \frac{2^N}{\delta_N} \sum_{|\beta| \leq N} |x^\beta|.$$

Hence (ii) ⇒ (i). The equivalence of (ii) and (iii) follows from the fact that each $\partial^\alpha(x^\beta f)$ is a linear combination of terms of the form $x^\delta \partial^\gamma f$ and vice versa, by the product rule. □

Proposition 1.14. *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then $f * g \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. By Proposition 1.4 (iii'), we have $f * g \in C^\infty(\mathbb{R}^n)$. Furthermore, since

$$1 + |x| \leq 1 + |x - y| + |y| \leq (1 + |x - y|)(1 + |y|),$$

we have for all order $N \in \mathbb{N}_0$ and multi-index $\alpha \in \mathbb{N}_0^n$ that

$$\begin{aligned}
(1 + |x|)^N |\partial^\alpha (f * g)(x)| &\leq \int_{\mathbb{R}^n} (1 + |x - y|)^N |\partial^\alpha (x - y)| (1 + |y|)^N |g(y)|\, dy \\
&\leq \|f\|_{(N,\alpha)} \|g\|_{(N+n+1,\alpha)} \int_{\mathbb{R}^n} (1 + |y|)^{-n-1}\, dy \\
&\leq \|f\|_{(N,\alpha)} \|g\|_{(N+n+1,\alpha)} \int_0^\infty \frac{C_n}{1 + r^2}\, dr < \infty,
\end{aligned}$$

where C_n is some constant depends only on the dimension n. □

Proposition 1.15. *$\mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ and in $C_0(\mathbb{R}^n)$.*

Proof. Since $\mathcal{S}(\mathbb{R}^n) \supset C_c^\infty(\mathbb{R}^n)$, the result follows from Propositions 1.7 and 1.8. □

2 Fourier Transform

2.1 Fourier Series

In this part, we study the periodic functions on $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{C}$ is said to be 2π-*periodic*, if

$$f(x + 2\pi\kappa) = f(x)$$

for all $x \in \mathbb{R}^n$ and all $\kappa \in \mathbb{Z}^n$. According to periodicity, every 2π-periodic function f is completely determined by its values on the cube $[0, 2\pi)^n$. Hence we may regard f as a function on the quotient space

$$\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n = \{x + 2\pi\mathbb{Z}^n : x \in \mathbb{R}^n\}.$$

We call $\mathbb{T}^n$ the n-dimensional torus. For measure-theoretic purposes, we identify $\mathbb{T}^n$ with the cube $Q = [0, 2\pi)^n$, and the Lebesgue measure on $\mathbb{T}^n$ is induced by Lebesgue measure on Q. In particular, $m(\mathbb{T}^n) = m(Q) = (2\pi)^n$. Functions on $\mathbb{T}^n$ maybe considered as periodic functions on $\mathbb{R}^n$ or as functions Q, depending on the context.

Theorem 2.1. *The functions $(e^{i\kappa\cdot x})_{\kappa\in\mathbb{Z}^n}$ form an orthogonal basis of $L^2(\mathbb{T}^n)$.*

Proof. Let $\mathcal{A}$ be the set of all finite linear combinations of $e^{i\kappa\cdot x}$. Then $\mathcal{A}$ is a self-adjoint algebra that separates points and vanishes at no points of $\mathbb{T}^n$. Since $\mathbb{T}^n$ is compact, by Stone-Weierstrass theorem, $\mathcal{A}$ is dense in $C(\mathbb{T}^n)$ in the supremum norm, and hence in L^2-norm. Since $C(\mathbb{T}^n)$ is dense in $L^2(\mathbb{T}^n)$, the result follows. □

The Fourier series of a periodic function is then defined by its expansion under the orthogonal basis.

Definition 2.2. If $f \in L^2(\mathbb{T}^n)$, we define its Fourier transform $\widehat{f} : \mathbb{Z}^n \to \mathbb{C}$ by

$$\widehat{f}(\kappa) = \frac{\langle f, e^{i\kappa\cdot x}\rangle_{L^2}}{\langle e^{i\kappa\cdot x}, e^{i\kappa\cdot x}\rangle_{L^2}} = \frac{1}{(2\pi)^n}\int_Q f(x)e^{-i\kappa\cdot x}\,dx, \tag{2.1}$$

and we call the series $\sum_{\kappa\in\mathbb{Z}^n} \widehat{f}(\kappa)e^{i\kappa\cdot x}$ the Fourier series of f.

Remark I. According to Theorem 2.1, the Fourier series of a function $f \in L^2(\mathbb{T}^n)$ converges to f in L^2. Consequently, we have the Parseval's equality:

$$\|\widehat{f}\|_{\ell^2}^2 := \sum_{\kappa\in\mathbb{Z}^n} |\widehat{f}(\kappa)|^2 = \frac{1}{(2\pi)^n}\|f\|_{L^2}^2.$$

Hence the Fourier transform $\mathcal{F}$ maps $L^2(\mathbb{T}^n)$ onto $\ell^2(\mathbb{Z}^n)$.

Remark II. In fact, the definition (2.1) of Fourier transform makes sense if $L^1(\mathbb{T}^n)$, and $|\widehat{f}(\kappa)| \le (2\pi)^{-n}\|f\|_{L^1}$. Hence the Fourier transform $\mathcal{F}$ is a bounded linear map from $L^1(\mathbb{T}^n)$ to $\ell^\infty(\mathbb{Z}^n)$.

Theorem 2.3 (Convolution Theorem)**.** *Let $f, g \in L^1(\mathbb{R}^n)$. Then*

$$\widehat{f * g} = (2\pi)^n \widehat{f}\,\widehat{g}.$$

Proof. By Young's convolution inequality [Proposition 1.3], $f * g \in L^1(\mathbb{T}^n)$. By Fubini's theorem,

$$\begin{aligned}\widehat{(f * g)}(\kappa) &= \frac{1}{(2\pi)^n}\int_Q\int_Q f(x-y)g(y)e^{-i\kappa\cdot x}\,dy\,dx = \int_Q\left(\frac{1}{(2\pi)^n}\int_Q f(x-y)e^{-i\kappa\cdot(x-y)}\,dx\right)g(y)e^{-i\kappa\cdot y}\,dy \\ &= \widehat{f}(\kappa)\int_Q g(y)e^{-i\kappa\cdot y}\,dy = (2\pi)^n\widehat{f}(\kappa)\,\widehat{g}(\kappa).\end{aligned}$$

Thus we finish the proof. □

2.2 Fourier Transform on $L^1(\mathbb{R}^n)$

Definition 2.4 (Fourier transform)**.** For $f \in L^1(\mathbb{R}^n)$, we define its *Fourier transform* by

$$(\mathcal{F}f)(\omega) = \widehat{f}(\omega) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-i\omega\cdot x}\, dx, \quad \omega \in \mathbb{R}^n,$$

and its *inverse Fourier transform* by

$$(\mathcal{F}^{-1}f)(x) = \check{f}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\omega) e^{i\omega\cdot x}\, d\omega, \quad x \in \mathbb{R}^n.$$

Remark. By definition, both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear operators. That is, for all $f, g \in L^1(\mathbb{R}^n)$ and $\alpha, \beta \in \mathbb{C}$,

$$\mathcal{F}(\alpha f + \beta g) = \alpha\mathcal{F}f + \beta\mathcal{F}g, \quad \mathcal{F}^{-1}(\alpha f + \beta g) = \alpha\mathcal{F}^{-1}f + \beta\mathcal{F}^{-1}g.$$

Also, we have $\check{f}(x) = \widehat{f}(-x)$. In the sequel, we first consider the Fourier transform.

Theorem 2.5 (Riemann-Lebesgue lemma)**.** *The Fourier transform $\mathcal{F}$ maps $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$.*

Proof. Fix $f \in L^1(\mathbb{R}^n)$. By definition, for all $\omega \in \mathbb{R}^n$,

$$|\widehat{f}(\omega)| \le (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(x)|\, dx.$$

Hence $\widehat{f}$ is bounded, and

$$\|\widehat{f}\|_\infty \le (2\pi)^{-n/2}\|f\|_{L^1}. \tag{2.2}$$

To show continuity of $\widehat{f}$, use dominated convergence theorem:

$$\begin{aligned}\lim_{h\to 0} f(\omega + h) - f(\omega) &= (2\pi)^{-n/2} \lim_{h\to 0} \int \underbrace{f(x)e^{-ix\cdot\omega}\left(e^{-ix\cdot h} - 1\right)}_{\textit{dominated by } 2|f|\in L^1(\mathbb{R}^n)} dx \\ &= (2\pi)^{-n/2} \int f(x)e^{-ix\cdot\omega} \lim_{h\to 0}\left(e^{-ix\cdot h} - 1\right) dx = 0.\end{aligned}$$

Hence $\widehat{f}$ is a bounded continuous function. It remains to show that $\widehat{f}(\omega) \to 0$ as $|\omega| \to \infty$. Note that

$$\begin{aligned}\widehat{f}(\omega) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\omega}\, dx = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f\left(x + \frac{\omega\pi}{|\omega|^2}\right) e^{-i\left(x+\frac{\omega\pi}{|\omega|^2}\right)\cdot\omega}\, dx \\ &= -(2\pi)^{-n/2} \int_{\mathbb{R}^n} f\left(x + \frac{\omega\pi}{|\omega|^2}\right) e^{-ix\cdot\omega}\, dx.\end{aligned}$$

By averaging,

$$\begin{aligned}|\widehat{f}(\omega)| &= \frac{(2\pi)^{-n/2}}{2}\left|\int_{\mathbb{R}^n}\left(f(x) - f\left(x + \frac{\omega\pi}{|\omega|^2}\right)\right)e^{-ix\cdot\omega}\, dx\right| \\ &\le \frac{(2\pi)^{-n/2}}{2}\int_{\mathbb{R}^n}\left|f(x) - f\left(x + \frac{\omega\pi}{|\omega|^2}\right)\right| dx \\ &= \frac{(2\pi)^{-n/2}}{2}\|f - \tau_h f\|_{L^1}, \quad \textit{where} \quad h = -\frac{\omega\pi}{|\omega|^2}.\end{aligned}$$

By translation continuity, the last display converges to 0 as $|\omega| \to \infty$. □

Remark. By (2.2), the Fourier transform $\mathcal{F} : L^1(\mathbb{R}^n) \to C_0(\mathbb{R}^n)$ is a bounded linear operator.

Proposition 2.6 (Properties of Fourier transform)**.** *Let $f, g \in L^1(\mathbb{R}^n)$.*

(i) $\int_{\mathbb{R}^n} \widehat{f}(x) g(x)\, dx = \int_{\mathbb{R}^n} f(x)\widehat{g}(x)\, dx$.

(ii) $\widehat{\overline{f}} = \overline{\check{f}}$, *and* $\check{\overline{f}} = \overline{\widehat{f}}$.

(iii) (Translation/Modulation) Let $\xi \in \mathbb{R}^n$. *Then* $\widehat{(\tau_\xi f)}(\omega) = e^{-i\omega\cdot\xi}\widehat{f}(\omega)$, *and* $\widehat{e^{i\xi\cdot x} f} = \tau_\xi \widehat{f}$.

(iv) (Linear transformation) If $T : \mathbb{R}^n \to \mathbb{R}^n$ *is an invertible linear transformation, and* $S = (T^*)^{-1}$ *is its inverse transpose, then*

$$\widehat{f \circ T} = |\det T|^{-1}\, \widehat{f} \circ S.$$

In particular, if T *is a rotation matrix, i.e.* $T^*T = TT^* = \mathrm{Id}$, *then* $\widehat{f \circ T} = \widehat{f} \circ T$*; if* $Tx = t^{-1}x$ *is a dilation, then* $\widehat{(f \circ T)}(\omega) = t^n \widehat{f}(t\omega)$.

Proof. (i) By Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^n} \widehat{f}(x) g(x)\, dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(\omega) e^{-i\omega\cdot x}\, d\omega \right) g(x)\, dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\omega) g(x) e^{-i\omega\cdot x}\, dx\, d\omega = \int_{\mathbb{R}^n} f(\omega)\widehat{g}(\omega)\, d\omega.\end{aligned}$$

(ii) We only prove the first identity (the second is similar):

$$\int_{\mathbb{R}^n} \overline{f(x)} e^{-i\omega\cdot x}\, dx = \overline{\int_{\mathbb{R}^n} f(x) e^{i\omega\cdot x}\, dx} = \overline{\check{f}(x)}.$$

(iii) By definition,

$$\widehat{(\tau_\xi f)}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x-\xi) e^{-i\omega\cdot x}\, dx = \frac{1}{(2\pi)^{n/2}} e^{-i\omega\cdot\xi} \int_{\mathbb{R}^n} f(x-\xi) e^{-i\omega\cdot(x-\xi)}\, dx = e^{i\omega\cdot\xi}\widehat{f}(\omega),$$

and

$$\widehat{(e^{i\xi\cdot x} f)}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\xi\cdot x} f(x) e^{-i\omega\cdot x}\, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i(\omega-\xi)\cdot x}\, dx = \widehat{f}(\omega - \xi).$$

(iv) By definition,

$$\begin{aligned}\widehat{(f\circ T)}(\omega) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(Tx) e^{i\omega\cdot x}\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \frac{1}{|\det T|} \int_{\mathbb{R}^n} f(y) e^{i\omega\cdot T^{-1}y}\, dy \\ &= \frac{1}{(2\pi)^{n/2}} \frac{1}{|\det T|} \int_{\mathbb{R}^n} f(y) e^{iS\omega\cdot y}\, dy = \frac{1}{|\det T|}\widehat{f}(S\omega).\end{aligned}$$

Thus we finish the proof. □

Remark. Let $\epsilon > 0$. Recall our notation that $\phi_\epsilon(x) = \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$, we have

$$\widehat{\phi_\epsilon}(\omega) = \widehat{\phi}(\epsilon\omega).$$

Moreover, if we let $g(x) = f(-x)$, then

$$\widehat{g}(x) = \widehat{f}(-x) = \check{f}(x).$$

Next we discuss the relation between Fourier transform and differentiation.

Proposition 2.7 (Differentiation)**.** *Let $k \in \mathbb{N}_0$ and $f \in L^1(\mathbb{R}^n)$.*

(i) If $x^\alpha f \in L^1(\mathbb{R}^n)$ for all multi-indices $|\alpha| \le k$, then $\widehat{f} \in C^k(\mathbb{R}^n)$, and

$$\partial^\alpha \widehat{f} = [(-ix)^\alpha f]^\wedge$$

(ii) If $f \in C^k(\mathbb{R}^n)$, $\partial^\alpha f \in L^1(\mathbb{R}^n)$ for all multi-indices $|\alpha| \le k$, and $\partial^\alpha f \in C_0(\mathbb{R}^n)$ for all $|\alpha| \le k-1$, then

$$\widehat{\partial^\alpha f}(\omega) = (i\omega)^\alpha \widehat{f}(\omega).$$

Proof. (i) Let $F(x,\omega) = f(x)e^{-i\omega\cdot x}$. Then

$$\frac{\partial F}{\partial \omega_j}(x,\omega) = -ix_j f(x) e^{-i\omega\cdot x}, \quad j = 1, 2, \cdots, n.$$

Fix $j \in \{1, 2, \cdots, n\}$. Note that when h is near 0, we have

$$\left| \frac{F(x,\omega + he_j) - F(x,\omega)}{h} \right| = \left| \frac{\mathrm{e}^{-ihx_j} - 1}{h} \right| |f(x)| \le 2|x_j f(x)|.$$

Since $x_j f \in L^1(\mathbb{R}^n)$, by dominated convergence theorem,

$$\begin{aligned}
\lim_{h\to 0} \frac{\widehat{f}(\omega + he_j) - \widehat{f}(\omega)}{h} &= \frac{1}{(2\pi)^{n/2}} \lim_{h\to 0} \int_{\mathbb{R}^n} \frac{F(x,\omega + he_j) - F(x,\omega)}{h}\, dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \lim_{h\to 0} \frac{F(x,\omega + he_j) - F(x,\omega)}{h}\, dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} -ix_j f(x) e^{-i\omega\cdot x}\, dx = \widehat{-ix_j f}.
\end{aligned}$$

(ii) Consider $|\alpha| = 1$. Since $\partial^\alpha f \in L^1(\mathbb{R}^n)$ and $f \in C_0(\mathbb{R}^n)$, use Fubini's theorem and integrate by parts:

$$\begin{aligned}
\widehat{\frac{\partial f}{\partial x_j}}(\omega) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_j}(x) e^{-i\omega\cdot x}\, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x_j}(x) e^{-i\omega_j x_j}\, dx_j \right) e^{-i\omega_{-j}\cdot x_{-j}}\, dx_{-j} \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(f(x) e^{-i\omega_j x_j} \Big|_{x_j=-\infty}^{x_j=\infty} + i\omega_j \int_{-\infty}^{\infty} f(x) e^{-i\omega_j x_j}\, dx_j \right) e^{-i\omega_{-j}\cdot x_{-j}}\, dx_{-j} \\
&= \frac{i\omega_j}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-i\omega\cdot x}\, dx = i\omega_j \widehat{f}(\omega).
\end{aligned}$$

Hence we prove the case $k = |\alpha| = 1$ for (i) and (ii). The general case follows from induction on $|\alpha|$. □

Theorem 2.8 (Convolution Theorem)**.** *Let $f, g \in L^1(\mathbb{R}^n)$. Then*

$$\widehat{f * g} = (2\pi)^{n/2} \widehat{f}\, \widehat{g}.$$

Proof. By Young's convolution inequality [Proposition 1.3], $f * g \in L^1(\mathbb{R}^n)$. By Fubini's theorem,

$$\begin{aligned}
\widehat{(f * g)}(\omega) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) g(y) e^{-i\omega\cdot x}\, dy\, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) e^{-i\omega\cdot(x-y)} g(y) e^{-i\omega\cdot y}\, dy\, dx \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x-y) e^{-i\omega\cdot(x-y)}\, dx \right) g(y) e^{-i\omega\cdot y}\, dy \\
&= \widehat{f}(\omega) \int_{\mathbb{R}^n} g(y) e^{-i\omega\cdot y}\, dy = (2\pi)^{n/2} \widehat{f}(\omega)\, \widehat{g}(\omega).
\end{aligned}$$

Thus we finish the proof. □

We compute the Fourier transform of a function.

Lemma 2.9. *Define the function* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *by* $\Phi(x) = e^{-\frac{|x|^2}{2}}$*. Then* $\Phi = \widehat{\Phi} = \check{\Phi}$*.*

Proof. For all $\omega \in \mathbb{R}^n$,

$$\begin{aligned}
\widehat{\Phi}(\omega) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} e^{-ix\cdot\omega}\, dx = \prod_{j=1}^{n} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x_j^2/2} e^{-ix_j\omega_j}\, dx_j \right) \\
&= \prod_{j=1}^{n} \left(\frac{e^{-\omega_j^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x_j+\mathrm{i}\omega_j)^2/2}\, dx_j \right) = \prod_{j=1}^{n} \left(\frac{e^{-\omega_j^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x_j^2/2}\, dx_j \right) \\
&= \prod_{j=1}^{n} e^{-\omega_j^2/2} = e^{-\frac{|\omega|^2}{2}}.
\end{aligned}$$

Hence $\widehat{\Phi} = \Phi$. The case $\check{\Phi} = \Phi$ is similar. ☐

Now we discuss how to recover a function f from its Fourier transform $\widehat{f}$.

Theorem 2.10 (Fourier inversion theorem)**.** *Let* $f \in L^1(\mathbb{R}^n)$*. If* $\widehat{f} \in L^1(\mathbb{R}^n)$*, then* $(\widehat{f})^{\vee} = f$ *a.e..*

Proof. We take the function Φ in Lemma 2.9. Consider the function

$$f^t(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \Phi(t\omega)\widehat{f}(\omega)e^{i\omega\cdot x}\, d\omega = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(t\omega) f(y) e^{i\omega\cdot(x-y)}\, dy\, d\omega.$$

Since $0 \le \Phi \le 1$ is bounded, $|\Phi(t\omega)\widehat{f}(\omega)| \le \widehat{f}(\omega)$. Since $\widehat{f} \in L^1(\mathbb{R}^n)$, by dominated convergence theorem,

$$\lim_{t\to 0} f^t(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \lim_{t\to 0} \Phi(t\omega)\widehat{f}(\omega)e^{i\omega\cdot x}\, d\omega = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{f}(\omega)e^{i\omega\cdot x}\, d\omega = (\widehat{f})^{\vee}(x), \quad \forall x \in \mathbb{R}^n.$$

On the other hand, if we show that $f^t \to f$ in L^1 as $t \to 0$, the result follows. By Fubini's theorem,

$$\begin{aligned}
f^t(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Phi(t\omega) f(y) e^{i\omega\cdot(x-y)}\, dy\, d\omega \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \Phi(t\omega) e^{i\omega\cdot(x-y)}\, d\omega \right) f(y)\, dy \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left(\frac{t^{-d}}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \Phi(\xi) f(y) e^{i\frac{\xi}{t}\cdot(x-y)}\, d\xi \right) dy \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} t^{-d}\Phi\left(\frac{x-y}{t}\right) f(y)\, dy = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \Phi_t\,(x-y)\, f(y)\, dy.
\end{aligned}$$

By Proposition 1.6, $\Phi_t * f \to (2\pi)^{n/2} f$ in L^1. Thus we complete the proof. ☐

Remark. We also have $\mathcal{F}\check{f} = f$ *a.e.* under the same assumption. To show this, let $g(x) = f(-x)$. Then

$$(\widehat{g})^{\vee}(x) = (\mathcal{F}^{-1}\check{f})(x) = (\mathcal{F}\check{f})(-x).$$

Since $(\widehat{g})^{\vee} = g$ *a.e.* and $g(x) = f(-x)$, the result follows.

Corollary 2.11. *If* $f \in L^1(\mathbb{R}^n)$ *and* $\widehat{f} = 0$ *a.e., then* $f = 0$ *a.e..*

Proof. Clearly $\widehat{f} = 0 \in L^1(\mathbb{R}^n)$. Then $f = (\widehat{f})^{\vee} = 0$. Here all equalities hold in a.e. sense. ☐

Remark. Also, if $f \in L^1(\mathbb{R}^n)$ and $\check{f} = 0$ *a.e.*, then $f = 0$ *a.e.*.

2.3 Fourier Transform on $L^2(\mathbb{R}^n)$

Theorem 2.12. *The Fourier transform $\mathcal{F}$ is an isomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ onto itself.*

Proof. Take $f \in \mathcal{S}(\mathbb{R}^n)$. By Proposition 1.13 (i), $x^\beta \partial^\alpha f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. By Proposition 2.7 (i), $\widehat{f} \in C^\infty(\mathbb{R}^n)$, and

$$\widehat{x^\beta \partial^\alpha f} = i^{|\beta|} \partial^\beta \widehat{(\partial^\alpha f)} = i^{|\alpha|+|\beta|} \partial^\beta (\omega^\alpha \widehat{f}).$$

Since $x^\beta \partial^\alpha f \in L^1(\mathbb{R}^n)$, we have $\partial^\beta(\omega^\alpha \widehat{f}) \in C_0(\mathbb{R}^n)$, which is bounded. By Proposition 1.13 (ii), $\widehat{f} \in \mathcal{S}(\mathbb{R}^n)$. Furthermore, since $\int_{\mathbb{R}^n} (1+|x|)^{-n-1}\, dx < \infty$, by Hölder's inequality,

$$\left\|\partial^\beta(\omega^\alpha \widehat{f})\right\|_\infty = \left\|\widehat{x^\beta \partial^\alpha f}\right\|_\infty \leq \left\|x^\beta \partial^\alpha f\right\|_{L^1} \leq C \left\|(1+|x|)^{n+1} x^\beta \partial^\alpha f\right\|_\infty \leq C \|f\|_{(|\beta|+n+1,\alpha)}.$$

Following the proof of Proposition 1.13, we have $\|\widehat{f}\|_{(N,\alpha)} \leq C_{N,\alpha} \sum_{|\gamma| \leq |\alpha|} \|f\|_{(N+n+1,\gamma)}$. Hence the Fourier transform $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R}^n)$ continuously into itself. On the other hand, since $\check{f}(x) = \widehat{f}(-x)$, the inverse Fourier transform $\mathcal{F}^{-1}$ also maps the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ into itself. By Fourier inversion theorem [Theorem 2.10], these maps are inverse to each other on $\mathcal{S}(\mathbb{R}^n)$. Hence we complete the proof. □

Theorem 2.13 (Plancherel)**.** *$\mathcal{F}$ extends from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to a unitary isomorphism on $L^2(\mathbb{R}^n)$.*

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^n)$, and let $h = \overline{\widehat{g}}$. Then

$$\widehat{h}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} \overline{\widehat{g}(x)} e^{-i\omega\cdot x}\, dx = \frac{1}{(2\pi)^{n/2}} \overline{\int_{\mathbb{R}^d} \widehat{g}(x) e^{i\omega\cdot x}\, dx} = \overline{(\widehat{g})^\vee(\omega)}$$

By Fourier inversion theorem, we have $\widehat{h} = \overline{g}$. Hence

$$\begin{aligned}
\langle f, g\rangle_{L^2} &= \int_{\mathbb{R}^d} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^d} f(x)\widehat{h}(x)\, dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)h(\omega)e^{-i\omega\cdot x}\, d\omega\, dx \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(x)e^{-i\omega\cdot x}\, dx\right) h(\omega)\, d\omega \qquad \text{(By Fubini's theorem)} \\
&= \int_{\mathbb{R}^d} \widehat{f}(\omega)h(\omega)\, d\omega = \int_{\mathbb{R}^d} \widehat{f}(\omega)\overline{\widehat{g}(\omega)}\, d\omega = \langle \widehat{f}, \widehat{g}\rangle_{L^2}.
\end{aligned}$$

Hence $\mathcal{F}|_{\mathcal{S}(\mathbb{R}^n)}$ preserves the L^2 inner product. Now for each $f \in L^2(\mathbb{R}^n)$, since $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we can take a sequence $f_k \in \mathcal{S}(\mathbb{R}^n)$ with $f_k \to f$ in L^2. Then $(\widehat{f_k})$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$:

$$\lim_{k,j\to\infty} \|\widehat{f_k} - \widehat{f_j}\|_{L^2} = \lim_{k,j\to\infty} \|\widehat{f_k - f_j}\|_{L^2} = \lim_{k,j\to\infty} \|f_k - f_j\|_{L^2} = 0.$$

This sequence converges to a limit $\widehat{f} = \mathcal{F}f \in L^2(\mathbb{R}^n)$. If $g_k \in \mathcal{S}(\mathbb{R}^n)$ with $g_k \to f$ in L^2, we have

$$\|\widehat{g} - \widehat{f}\|_{L^2} = \lim_{k\to\infty} \|\widehat{g}_k - \widehat{f}_k\|_{L^2} = \lim_{k\to\infty} \|g_k - f_k\|_{L^2} \leq \lim_{k\to\infty} \|g_k - f\|_{L^2} + \lim_{k\to\infty} \|f - f_k\|_{L^2} = 0.$$

Hence the limit does not depend on the choice of the sequence (f_k), and the transform $\widehat{f} = \mathcal{F}f$ is well-defined. Furthermore, for all $f, g \in L^2(\mathbb{R}^n)$, we have

$$\langle f, g\rangle_{L^2} = \langle \widehat{f}, \widehat{g}\rangle_{L^2}.$$

Hence $\mathcal{F}$ is a unitary isomorphism on $L^2(\mathbb{R}^n)$. □

Remark. Likewise, $\mathcal{F}^{-1}$ also extends from $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ to a unitary isomorphism on $L^2(\mathbb{R}^n)$.

Corollary 2.14. *Let $f \in L^2(\mathbb{R}^n)$. Then $(\widehat{f})^\vee = f$.*

Proof. Take a sequence $f_k \in \mathcal{S}(\mathbb{R}^n)$ with $f_k \to f$ in L^2. Then $\widehat{f_k} \to \widehat{f}$ in L^2, and $f_k = (\widehat{f_k})^\vee \to (\widehat{f})^\vee$ in L^2. $\square$

Also, we have an explicit formula for Fourier transform in L^2.

Corollary 2.15. *Let $f \in L^2(\mathbb{R}^n)$. Then*

$$\widehat{f}(\omega) = \frac{1}{(2\pi)^{n/2}} \lim_{N\to\infty} \int_{|x|\le N} f(x)e^{-i\omega\cdot x}\, dx,$$

where the limit is in L^2 sense.

Proof. We choose $f_N = f\chi_{\{|x|\le N\}}$, which is in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by Cauchy-Schwarz inequality, and converges in L^2 to f as $N \to \infty$, by monotone convergence theorem. By Plancherel theorem, $\widehat{f_N} \to \widehat{f}$ in L^2. $\square$

Finally we introduce the convolution theorem for L^2-functions.

Proposition 2.16. *If $f, g \in L^2(\mathbb{R}^n)$, then $(\widehat{f}\,\widehat{g})^\vee = (2\pi)^{-n/2}(f * g)$.*

Proof. By Plancherel's theorem and Hölder's inequality, we have $\widehat{f}, \widehat{g} \in L^2(\mathbb{R}^n)$, and $\widehat{f}\,\widehat{g} \in L^1(\mathbb{R}^n)$. We fix $x \in \mathbb{R}^n$, and set $h_x(y) = \overline{g(x-y)}$. Then

$$\widehat{h_x}(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \overline{g(x-y)}e^{-i\omega\cdot y}\, dy = \frac{1}{(2\pi)^{n/2}} \overline{\int_{\mathbb{R}^n} g(x-y)e^{-i\omega\cdot(x-y)}\, dy\, e^{i\omega\cdot x}} = \overline{\widehat{g}(\omega)}e^{-i\omega\cdot x}.$$

Since $\mathcal{F}$ is unitary in $L^2(\mathbb{R}^n)$,

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\overline{h_x(y)}\, dy = \int_{\mathbb{R}^n} \widehat{f}(\omega)\overline{\widehat{h_x}(\omega)}\, d\omega = \int_{\mathbb{R}^n} \widehat{f}(\omega)\widehat{g}(\omega)e^{i\omega x}\, d\omega = (2\pi)^{n/2}(\widehat{f}\,\widehat{g})^\vee(x).$$

Thus we complete the proof. $\square$

By Fourier inversion theorem and linearity of Laplacian operator,

$$\Delta f(x) = \Delta \int_{\mathbb{R}^n} \frac{\widehat{f}(\omega)}{(2\pi)^{n/2}} e^{i\omega\cdot x}\, d\omega = \int_{\mathbb{R}^n} \frac{\widehat{f}(\omega)}{(2\pi)^{n/2}} \Delta e^{i\omega\cdot x}\, d\omega = -\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |\omega|^2 \widehat{f}(\omega)e^{i\omega\cdot x}\, d\omega$$

By taking the Fourier transform on both sides, we have

$$\widehat{\Delta f}(\omega) = -|\omega|^2 \widehat{f}(\omega).$$

2.4 Fourier Transform of Radial Functions and Hankel Transform

Bessel functions. Consider the Bessel's differential equation about function $y(z)$:

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0. \tag{2.3}$$

The *Bessel function of the first kind* of order $\nu \in \mathbb{C}$ solves this equation:

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\nu+m+1)} \left(\frac{z}{2}\right)^{2m+\nu}, \quad z \in \mathbb{C}\backslash\{0\},$$

where the power in this definition is given by $z^\nu = e^{\nu \log z}$, where $\log z$ is chosen to be the principal branch of the logarithm, i.e. $-\pi < \arg(z) \leq \pi$. The Bessel function $J_\nu(z)$ is holomorphic in $\mathbb{C}\backslash(-\infty, 0]$ for every $\nu \in \mathbb{C}$.

- When $\nu \notin \mathbb{Z}$, the Bessel functions $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent, and the general solution of the Bessel's equation is

$$y(z) = \gamma_1 J_\nu(z) + \gamma_2 J_{-\nu}(z), \quad \gamma_1, \gamma_2 \in \mathbb{C}.$$

- When $\nu = n \in \mathbb{Z}$, the Bessel function J_n has an analytic extension to $\mathbb{C}$. Furthermore, using the property that $1/\Gamma(-n) = 0$ for nonnegative integers $n = 0, 1, 2, \cdots$, we have

$$J_{-n}(z) = (-1)^n J_n(z), \quad n \in \mathbb{N}_0.$$

- To get a solution of (2.3) when $\nu = n \in \mathbb{Z}$ that is linearly independent of from $J_{\pm\nu}$, we introduce the*Bessel function of the second kind* of order $\nu \in \mathbb{C}$, which is defined as

$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, \quad \nu \notin \mathbb{Z}, \quad \textit{and} \quad Y_n(z) = \lim_{\nu \notin \mathbb{Z},\, \nu \to n} Y_\nu(z), \quad n \in \mathbb{Z}.$$

  The Bessel function $Y_n(z)$ solves (2.3) when $\nu = n \in \mathbb{Z}$.

Proposition 2.17. *Let $\nu \in \mathbb{C}$, and let $J_\nu(z)$ be the Bessel function of the first kind.*

(i) The following recursive formulae hold:

$$J_{\nu-1}(z) = \frac{dJ_\nu}{dz} + \frac{\nu}{z}J_\nu(z), \quad \text{and} \quad J_{\nu+1}(z) = -\frac{dJ_\nu}{dz} + \frac{\nu}{z}J_\nu(z).$$

(ii) In particular,

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin(z), \quad \text{and} \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos(z).$$

Remark. Combining the two assertions, one can recurrently derive Bessel functions of half integer orders.

Proof. (i) The first formula follows from the following identity:

$$\frac{d}{dz}\left[z^\nu J_\nu(z)\right] = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+2\nu)}{\Gamma(m+1)\Gamma(\nu+m+1)} \frac{z^{2m+2\nu-1}}{2^{2m+\nu}} = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(\nu+m)} \frac{z^{2m+2\nu-1}}{2^{2m+\nu-1}} = z^\nu J_{\nu-1}(z).$$

Similarly, the second formula follows from the following identity:

$$\begin{aligned}\frac{d}{dz}\left[z^{-\nu} J_\nu(z)\right] &= \sum_{m=0}^{\infty} \frac{(-1)^m(2m)}{\Gamma(m+1)\Gamma(\nu+m+1)} \frac{z^{2m-1}}{2^{2m+\nu}} = \sum_{m=1}^{\infty} \frac{(-1)^m}{\Gamma(m)\Gamma(\nu+m+1)} \frac{z^{2m-1}}{2^{2m+\nu-1}} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{\Gamma(m+1)\Gamma(\nu+m+2)} \frac{z^{2m+1}}{2^{2m+\nu+1}} = -z^{-\nu} J_{\nu+1}(z).\end{aligned}$$

(ii) Note that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Then

$$J_{\frac{1}{2}}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+\frac{3}{2})} \left(\frac{z}{2}\right)^{2m+\frac{1}{2}} = \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \left(\frac{z}{2}\right)^{2m+1}$$
$$= \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!\sqrt{\pi}} z^{2m+1} = \sqrt{\frac{2}{\pi z}} \sin(z),$$

and

$$J_{-\frac{1}{2}}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+\frac{1}{2})} \left(\frac{z}{2}\right)^{2m-\frac{1}{2}} = \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \left(\frac{z}{2}\right)^{2m}$$
$$= \sqrt{\frac{2}{z}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!\sqrt{\pi}} z^{2m} = \sqrt{\frac{2}{\pi z}} \cos(z).$$

Therefore we complete the proof. □

The Bessel functions are related to the integral of plane wave functions on the sphere.

Proposition 2.18 (Sphere integral form of the Bessel functions of the first kind)**.** *Let $n \geq 2$, and denote by $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ the unit sphere in $\mathbb{R}^n$. Then*

$$\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega) = (2\pi)^{\frac{n}{2}} |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(|x|). \tag{2.4}$$

The proof of this result requires some technical lemmata. We first introduce a type of special integrals.

Lemma 2.19. *For each $n, m \in \mathbb{N}_0$,*

$$\int_0^{\pi} \sin^n\theta \cos^{2m}\theta\, d\theta = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(m+\frac{n}{2}+1\right)}.$$

In particular,

$$\int_0^{\pi} \sin^n\theta\, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

Proof. (i) We begin from the second integral. Let $I_n = \int_0^{\pi} \sin^n\theta\, d\theta$. To begin with, we have $I_0 = \pi$ and $I_1 = 2$. For $n \geq 2$, compute I_n recurrently:

$$I_n = -\int_0^{\pi} \sin^{n-1}\theta\, d\cos\theta = \int_0^{\pi} (n-1)\sin^{n-2}\theta\cos^2\theta\, d\theta = (n-1)(I_{n-2} - I_n).$$

Hence $I_n = \frac{n-1}{n} I_{n-2}$. By induction, for any $n \in \mathbb{N}_0$,

$$I_{2k+1} = \frac{2k}{2k+1} \cdot \frac{2k-2}{2k-1} \cdot \cdots \cdot \frac{2}{3} \cdot I_1 = \frac{\Gamma(k+1)\sqrt{\pi}}{\Gamma\left(k+\frac{3}{2}\right)},$$

and

$$I_{2k} = \frac{2k-1}{2k} \cdot \frac{2k-3}{2k-2} \cdot \cdots \cdot \frac{1}{2} \cdot I_0 = \frac{\Gamma\left(k+\frac{1}{2}\right)\sqrt{\pi}}{\Gamma(k+1)}.$$

The first result is obtained by summarizing the last two identities.

(ii) Let $I_{n,m} = \int_0^\pi \sin^n\theta \cos^{2m}\theta\, d\theta$. Then

$$\begin{aligned}
I_{n,m} &= \int_0^\pi \sin^n\theta \cos^{2m-1}\theta\, d\sin\theta = -\int_0^\pi \sin\theta\, d\left(\sin^n\theta\cos^{2m-1}\theta\right) \\
&= -n\int_0^\pi \sin^n\theta\cos^{2m}\theta\, d\theta + (2m-1)\int_0^\pi \sin^{n+2}\theta\cos^{2m-2}\theta\, d\theta \\
&= -nI_{n,m} + (2m-1)(I_{n,m-1} - I_{n,m}) = (1-2m-n)I_{n,m} + (2m-1)I_{n,m-1}.
\end{aligned}$$

Hence $I_{n,m} = \frac{2m-1}{2m+n} I_{n,m-1}$. By induction,

$$\begin{aligned}
I_{n,m} &= \frac{2m-1}{2m+n}\cdot\frac{2m-3}{2m+n-2}\cdot\cdots\cdot\frac{1}{n+2}\cdot I_{n,0} \\
&= \frac{2^m\Gamma\left(m+\frac{1}{2}\right)/\sqrt{\pi}}{2^m\Gamma\left(m+\frac{n}{2}+1\right)/\Gamma\left(\frac{n}{2}+1\right)}\frac{\Gamma\left(\frac{n+1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(m+\frac{n}{2}+1\right)}.
\end{aligned}$$

Therefore the first result holds. □

Lemma 2.20. *Let $n \geq 2$. The surface area of unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is $\sigma_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.*

Proof. Using the spherical coordinates, and by Lemma 2.19, we have

$$\sigma_{n-1} = \int_{S^{n-1}} dS(x) = \int_0^\pi \sigma_{n-2}\sin^{n-2}\theta\, d\theta = \frac{\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)}\sigma_{n-2}.$$

Since $\sigma_1 = 2\pi$ and $\Gamma(1) = 1$, the result follows by induction. □

Proof of Proposition 2.18. Let $r = |x|$. Since $\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega)$ is radial about x, we take $x = (r, 0, \cdots, 0)$:

$$\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega) = \int_{S^{n-1}} e^{ir\omega_1}\, dS(\omega). \tag{2.5}$$

For $\omega \in S^{n-1}$, let $\theta = \arccos(\langle\omega, e_1\rangle)$, where $e_1 = (1, 0, \cdots, 0)$. Then $\cos\theta = \omega_1$, and $\sin\theta = \sqrt{\omega_2^2 + \cdots + \omega_n^2}$. Switching to the spherical coordinates, we have

$$\begin{aligned}
\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega) &= \int_{S^{n-1}} e^{ir\omega_1}\, dS(\omega) = \int_0^\pi e^{ir\cos\theta}\sigma_{n-2}\sin^{n-2}\theta\, d\theta \\
&= \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\pi e^{ir\cos\theta}\sin^{n-2}\theta\, d\theta.
\end{aligned} \tag{2.6}$$

We compute the last integral by expanding the exponent and integrating term by term:

$$\begin{aligned}
\int_0^\pi e^{ir\cos\theta}\sin^{n-2}\theta\, d\theta &= \sum_{k=0}^\infty \frac{(ir)^k}{k!}\int_0^\pi \cos^k\theta\sin^{n-2}\theta\, d\theta = \sum_{m=0}^\infty \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(m+\frac{n}{2}\right)}\frac{(ir)^{2m}}{(2m)!} \\
&= \sum_{m=0}^\infty \frac{(2m-1)!!\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{2^m\Gamma\left(m+\frac{n}{2}\right)}\frac{(ir)^{2m}}{(2m)!} = \sum_{m=0}^\infty \frac{(-1)^m\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{m!\,\Gamma\left(m+\frac{n}{2}\right)}\left(\frac{r}{2}\right)^{2m},
\end{aligned} \tag{2.7}$$

where the odd terms vanishes by symmetry on $[0, \pi]$, and the even terms follow from Lemma 2.19. Combining (2.6) and (2.7), we obtain

$$\int_{S^{n-1}} e^{i\omega\cdot x}\, dS(\omega) = 2\pi^{\frac{n}{2}}\sum_{m=0}^\infty \frac{(-1)^m}{m!\,\Gamma\left(m+\frac{n}{2}\right)}\left(\frac{r}{2}\right)^{2m} = (2\pi)^{\frac{n}{2}} r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r).$$

Thus we complete the proof. □

We now turn to the Laplace transforms of some specific functions involving Bessel functions.

Proposition 2.21. *For every $\nu > -1$ and $r > 0$,*

$$\int_0^\infty J_\nu(x)x^{\nu+1}e^{-rx}\,dx = \frac{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)r}{\sqrt{\pi}(1+r^2)^{\nu+\frac{3}{2}}}. \tag{2.8}$$

Proof. For $0 < r < 1$ and $\mu > 0$, the Taylor series of $(1+r)^{-\mu}$ is

$$(1+r)^{-\mu} = \sum_{m=0}^\infty \frac{(-1)^m\Gamma(\mu+m)}{m!\,\Gamma(\mu)}r^m.$$

Replacing r by $1/r^2$, we have

$$\frac{r^{2\mu}}{(1+r^2)^\mu} = \sum_{m=0}^\infty \frac{(-1)^m\Gamma(\mu+m)}{m!\,\Gamma(\mu)}r^{-2m}, \quad r > 1.$$

Hence the right hand side of (2.8) is

$$\begin{aligned}\frac{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)r}{\sqrt{\pi}(1+r^2)^{\nu+\frac{3}{2}}} &= \frac{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)r^{-2\nu-2}}{\sqrt{\pi}}\sum_{m=0}^\infty \frac{(-1)^m\Gamma(\nu+\frac{3}{2}+m)}{m!\,\Gamma\left(\nu+\frac{3}{2}\right)}r^{-2m}\\ &= \sum_{m=0}^\infty \frac{(-1)^m 2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}+m\right)}{\Gamma(m+1)\sqrt{\pi}}r^{-2m-2\nu-2}\\ &= \sum_{m=0}^\infty \frac{(-1)^m\Gamma\left(2\nu+2m+2\right)}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)}r^{-2m-2\nu-2},\end{aligned} \tag{2.9}$$

where the last equality follows from Legendre's duplication formula. Now we turn to the integral. By Sterling's formula, there exists a constant c_ν depending only on $\nu > -1$ such that $\Gamma\left(\nu+m+1\right) \geq \frac{m!}{c_\nu}$. Then

$$\begin{aligned}\sum_{m=1}^\infty \frac{x^{2m+2\nu+1}e^{-rx}}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)} &\leq \frac{c_\nu}{2^\nu}x^{2\nu+1}e^{-rx}\sum_{m=1}^\infty \frac{x^{2m}}{(2^m m!)^2}\\ &\leq \frac{c_\nu}{2^\nu}x^{2\nu+1}e^{-rx}\sum_{m=1}^\infty \frac{x^{2m}}{(2m)!} \leq \frac{c_\nu}{2^\nu}x^{2\nu+1}e^{-(r-1)x},\end{aligned}$$

which is absolutely integrable. Using dominated convergence theorem, we can interchange infinite summation and integral in the left hand side of (2.8):

$$\begin{aligned}\int_0^\infty J_\nu(x)x^{\nu+1}e^{-rx}\,dx &= \sum_{m=1}^\infty \frac{(-1)^m}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)}\int_0^\infty x^{2m+2\nu+1}e^{-rx}\,dx\\ &= \sum_{m=1}^\infty \frac{(-1)^m r^{-2m-2\nu-2}}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)}\int_0^\infty y^{-2m-2\nu-1}e^{-y}\,dy\\ &= \sum_{m=0}^\infty \frac{(-1)^m\Gamma\left(2\nu+2m+2\right)}{2^{2m+\nu}\Gamma(m+1)\Gamma\left(\nu+m+1\right)}r^{-2m-2\nu-2},\end{aligned}$$

which is consistent with (2.9). Hence the identity (2.8) holds for $r > 1$. Finally, since both sides of (2.9) is analytic in the region $\mathrm{Re}(r) > 0$ and $|\mathrm{Im}(r)| < 1$, the case $0 < r \leq 1$ follows from analytic continuation. ☐

Now we study the Fourier transform of radial functions on $\mathbb{R}^n$. A function $F : \mathbb{R}^n \to \mathbb{C}$ is said to be radial, if there exists a function f such that $F(x) = f(|x|)$ for all $x \in \mathbb{R}^n$.

Definition 2.22 (Hankel transform)**.** Let $\nu \geq -\frac{1}{2}$. We define the *Hankel transform of order* ν of a function $f \in L^2((0,\infty), r\,dr)$ by

$$(H_\nu f)(\lambda) = \int_0^\infty r f(r) J_\nu(\lambda r)\,dr, \quad \lambda > 0.$$

The Hankel transform of order $\frac{n}{2} - 1$ is related to the Fourier transform of radial functions in $\mathbb{R}^n$.

Theorem 2.23. *Let $F \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be a radial function, i.e. $F(x) = f(|x|)$ for $x \in \mathbb{R}^n$. Then the Fourier transform $\widehat{F}$ is also radial, i.e. $\widehat{F}(\omega) = \phi(|\omega|)$, with*

$$\phi(\lambda) = \lambda^{1-\frac{n}{2}} \int_0^\infty r^{\frac{n}{2}} f(r) J_{\frac{n}{2}-1}(\lambda r)\,dr$$

In other words, $|\omega|^{\frac{n}{2}-1}\widehat{F}(\omega)$ coincides the Hankel transform of order $\frac{n}{2} - 1$ of $r^{\frac{n}{2}-1}f(r)$

Proof. For the case $n = 1$, we have $J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos(z)$ by Proposition 2.17. Since $F : \mathbb{R} \to \mathbb{C}$ is even,

$$\begin{aligned}\widehat{F}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F(x) e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty F(x)\left(\cos(\omega x) - i\sin(\omega x)\right)dx \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(r)\cos(|\omega| r)\,dr = |\omega|^{\frac{1}{2}} \int_0^\infty \sqrt{r} f(r) J_{-\frac{1}{2}}(|\omega| r)\,dr.\end{aligned}$$

For the case $n \geq 2$, we switch to sphere coordinates and use Proposition 2.18:

$$\begin{aligned}\widehat{F}(\omega) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} F(x) e^{-i\omega\cdot x}\,dx = (2\pi)^{-\frac{n}{2}} \int_0^\infty r^{n-1} \int_{S^{n-1}} f(r|x|) e^{-ir\omega\cdot x}\,dS(x)\,dr \\ &= (2\pi)^{-\frac{n}{2}} \int_0^\infty r^{n-1} f(r) \left(\int_{S^{n-1}} e^{-ir\omega\cdot x}\,dS(x)\right) dr \\ &= (2\pi)^{-\frac{n}{2}} \int_0^\infty r^{n-1} f(r) \cdot (2\pi)^{\frac{n}{2}} (r|\omega|)^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r|\omega|)\,dr \\ &= |\omega|^{1-\frac{n}{2}} \int_0^\infty r^{\frac{n}{2}} f(r) J_{\frac{n}{2}-1}(r|\omega|)\,dr.\end{aligned}$$

Then we conclude the proof. □

Remark. In particular, taking $n = 2$, we know that the Hankel transform of order 0 coincides the Fourier transformation of radial function in $\mathbb{R}^2$.

2.5 Application in Partial Differential Equations

Fourier transform and differential operators. Consider the Laplacian operator:

$$\Delta : C^2(\mathbb{R}^n) \to C(\mathbb{R}^n), \qquad \Delta f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}.$$

For the plane wave function $f(x) = e^{i\omega\cdot x}$, we have

$$\Delta e^{i\omega\cdot x} = \sum_{j=1}^{n} (i\omega_j)^2 e^{i\omega\cdot x} = -|\omega|^2 e^{i\omega\cdot x}.$$

In other words, the function $e^{i\omega\cdot x}$ is an eigenfunction of Δ, with eigenvalue $-|\omega|^2$. Furthermore, under regularity conditions [See Proposition 2.7], we have

$$\widehat{\Delta f}(\omega) = \sum_{j=1}^{n} (i\omega_j)^2 \widehat{f}(\omega) = -|\omega|^2 \widehat{f}(\omega).$$

This identity shows that *the Fourier transform diagonalizes the Laplacian* Δ. In other words, the Laplacian is nothing more than an explicit multiplier when viewed using the Fourier transform.

Example 2.24 (Heat equation with Dirichlet boundary condition)**.** Consider the heat equation about the time-varying function $u(x,t)$, which is defined on $\mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{cases} u_t = \Delta_x u & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \times \{t=0\}, \\ \lim_{|x|\to\infty} u(x,t) = 0 & \text{for } t \in [0,\infty), \end{cases} \tag{2.10}$$

where the initial function $f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. We let $\widehat{u}(\omega,t) = \int_{\mathbb{R}^n} u(x,t) e^{-i\omega\cdot x}\, dx$ be the Fourier transform of u with respect to x. Applying Fourier transform on both the heat equation and the initial condition, we get the initial value problem:

$$\begin{cases} \widehat{u}_t = -|\omega|^2 \widehat{u}, \\ \widehat{u}(\omega,0) = \widehat{f}(\omega). \end{cases}$$

The solution of this problem is given by $\widehat{u}(\omega,t) = \widehat{f}(\omega) e^{-|\omega|^2 t}$. To recover u, we employ the inverse Fourier transform and convolution theorem [Theorem 2.8]:

$$u(x,t) = \mathcal{F}^{-1}\left(\widehat{f}(\omega) e^{-|\omega|^2 t}\right) = (2\pi)^{-n/2} f * \mathcal{F}^{-1}(e^{-|\omega|^2 t}).$$

It remains to compute the inverse Fourier transform of $e^{-|\omega|^2 x}$:

$$\begin{aligned} \mathcal{F}^{-1}(e^{-|\omega|^2 t})(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|\omega|^2 t} e^{i\omega\cdot x}\, d\omega = \prod_{j=1}^{n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\omega_j^2 t + i\omega_j x_j}\, d\omega_j \\ &= \prod_{j=1}^{n} e^{-\frac{x_j^2}{4t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\left(\omega_j\sqrt{t} - \frac{ix_j}{2\sqrt{t}}\right)^2} d\omega_j = \prod_{j=1}^{n} \frac{1}{\sqrt{2t}} e^{-\frac{x_j^2}{4t}} = \frac{1}{(2t)^{n/2}} e^{-\frac{|x|^2}{4t}}. \end{aligned}$$

Hence the solution of problem (2.10) is given by

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|y-x|^2}{4t}} f(y)\, dy. \qquad \square$$

Remark. We write the heat kernel by

$$\Phi_t(x) = \begin{cases} \delta(x), & t = 0, \\ (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}, & t > 0. \end{cases}$$

Then the solution of problem (2.10) can be represented as $u = \Phi_t * f$.

Example 2.25 (Heat equation with a source)**.** Consider the heat equation about the time-varying function $u(x,t)$, which is defined on $\mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{cases} u_t = \Delta_x u + S(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \times \{t = 0\}, \\ \lim_{|x|\to\infty} u(x,t) = 0 & \text{for } t \in [0,\infty), \end{cases} \tag{2.11}$$

where the source $S(x,t) \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$ for every t, and the initial function $f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. Similar to the case without the source $S(x,t)$, we apply Fourier transform on both the equation and the initial condition to get an initial value problem:

$$\begin{cases} \widehat{u}_t = -|\omega|^2 \widehat{u} + \widehat{S}(\omega,t), \\ \widehat{u}(\omega,0) = \widehat{f}(\omega). \end{cases}$$

We solve this problem by multiplying by a factor $e^{|\omega|^2 t}$:

$$\begin{aligned} \frac{\partial}{\partial t}\left(e^{|\omega|^2 t}\widehat{u}\right) &= e^{|\omega|^2 t}\left(\widehat{u}_t + |\omega|^2 \widehat{u}\right) = e^{|\omega|^2 t}\widehat{S}(\omega,t), \\ e^{|\omega|^2 t}\widehat{u}(\omega,t) &= \widehat{f}(\omega) + \int_0^t e^{|\omega|^2 \tau}\widehat{S}(\omega,\tau)\,d\tau, \\ \widehat{u}(\omega,t) &= e^{-|\omega|^2 t}\widehat{f}(\omega) + \int_0^t e^{-|\omega|^2(t-\tau)}\widehat{S}(\omega,\tau)\,d\tau. \end{aligned}$$

Applying inverse Fourier transform, we obtain the solution of (2.11):

$$u(x,t) = \int_{\mathbb{R}^n} \Phi_t(x-y) f(y)\,dt + \int_0^t \int_{\mathbb{R}^n} \Phi_{t-\tau}(x-y) S(y,t)\,dy\,d\tau. \qquad \square$$

Example 2.26 (Laplace equation in the upper half space)**.** Consider the Laplace equation about the function $u(x,y)$ in the upper half space $\mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \times \{t = 0\}, \\ \lim_{|x|\to\infty} u(x,y) = 0 & \text{and } \lim_{y\to\infty} u(x,y) = 0, \end{cases} \tag{2.12}$$

where the function $f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. We write the Laplace equation as $u_{yy} = \Delta_x u$, and apply Fourier transform on the variable x. Then we get the following initial value problem:

$$\begin{cases} \widehat{u}_{yy} = |\omega|^2 \widehat{u}, \\ \widehat{u}(\omega,0) = \widehat{f}(\omega), \\ \lim_{y\to\infty} \widehat{u}(\omega,y) = 0 \end{cases}$$

Since u is vanishing as $y \to \infty$, the solution to this problem is

$$\widehat{u}(\omega, y) = e^{-|\omega| y} \widehat{f}(\omega).$$

Hence the solution to (2.12) is

$$u(x, y) = \frac{1}{(2\pi)^{n/2}} \cdot \mathcal{F}^{-1}(e^{-|\omega| y}) * f.$$

We compute inverse Fourier transform of $e^{-|\omega| y}$:

$$\begin{aligned}
\mathcal{F}^{-1}(e^{-|\omega| y}) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-|\omega| y} e^{i\omega \cdot x} \, d\omega = \frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_{\partial B(x,r)} e^{-|\omega| y} e^{i\omega \cdot x} \, dS(\omega) \, dr \\
&= \frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_{S^{n-1}} e^{-ry} e^{ir\xi \cdot x} r^{n-1} \, dS(\xi) \, dr \\
&= \int_0^\infty r^{\frac{n}{2}} e^{-ry} |x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r|x|) \, dr && \text{(By Proposition 2.18)} \\
&= |x|^{-n} \int_0^\infty \rho^{\frac{n}{2}} e^{-\rho \frac{y}{|x|}} J_{\frac{n}{2}-1}(\rho) \, d\rho = \frac{2^{\frac{n}{2}} \Gamma\left(\frac{n+1}{2}\right) y}{\sqrt{\pi} \left(|x|^2 + y^2\right)^{\frac{n+1}{2}}}. && \text{(By Proposition 2.21)}
\end{aligned}$$

Then

$$u(x, y) = \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \int_{\mathbb{R}^n} \frac{y}{\left(|x - z|^2 + y^2\right)^{\frac{n+1}{2}}} f(z) \, dz. \qquad \square$$

Remark. We define the *Poisson kernel* by

$$P(x, y) = c_n \frac{y}{\left(|x|^2 + y^2\right)^{\frac{n+1}{2}}}, \quad \text{where } c_n = \pi^{-\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right).$$

Then the solution of problem (2.10) can be represented as $u(\cdot, y) = P(\cdot, y) * f$.

Example 2.27 (Wave equation with Dirichlet boundary condition)**.** Consider the wave equation about the time-varying function $u(x, t)$, which is defined on $\mathbb{R}^n \times \mathbb{R}_+$:

$$\begin{cases} u_{tt} = \Delta_x u & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x, 0) = f(x), \ u_t(x, 0) = g(x) & \text{on } \mathbb{R}^n \times \{y = 0\}, \\ \lim_{|x| \to \infty} u(x, t) = 0 & \text{for } \ t \in [0, \infty). \end{cases} \tag{2.13}$$

where the functions $f, g \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. Applying Fourier transform with respect to the variable $x \in \mathbb{R}^n$, we get the initial value problem

$$\begin{cases} \widehat{u}_{tt} = -|\omega|^2 \widehat{u}, \\ \widehat{u}(\omega, 0) = \widehat{f}(\omega), \quad \widehat{u}_t(\omega, 0) = \widehat{g}(\omega). \end{cases}$$

The solution of this initial value problem is

$$\widehat{u}(\omega, t) = \widehat{f}(\omega) \cos(|\omega| t) + \widehat{g}(\omega) \frac{\sin(|\omega| t)}{|\omega|}.$$

We write $R(x, t) = \frac{1}{(2\pi)^{n/2}} \mathcal{F}^{-1}\left(\frac{\sin(|\omega| t)}{|\omega|}\right)$. By convolution theorem, the solution to problem 2.13 is

$$u(\cdot, t) = \frac{\partial}{\partial t} \left(R(\cdot, t) * f\right) + R(\cdot, t) * g. \qquad \square$$

Example 2.28 (Transport equation)**.** Consider the following transport equation with constant coefficients:

$$\begin{cases} u_t - b \cdot \nabla_x u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R}^n \times \{t = 0\}, \\ \lim_{|x|\to\infty} u(x,t) = 0 & \text{for } \ t \in [0, \infty), \end{cases} \tag{2.14}$$

where the velocity $b \in \mathbb{R}^n$ is a constant vector, and $f \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$.

Solution. We apply Fourier transform with respect to the variable x:

$$\begin{cases} \widehat{u}_t = ib \cdot \omega \widehat{u}, \\ \widehat{u}(\omega, 0) = \widehat{f}(\omega). \end{cases}$$

Then $\widehat{u}(\omega, t) = e^{itb\cdot\omega}\widehat{f}(\omega)$, and the solution to problem (2.14) is

$$u(x,t) = \mathcal{F}^{-1}\left[e^{itb\cdot\omega}\widehat{f}(\omega)\right] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{f}(\omega) e^{i(x+tb)\cdot\omega}\, d\omega = f(x + tb).$$ □

Example 2.29 (Linearized Korteweg-De Vries equation)**.** Consider the equation about $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}$.

$$\begin{cases} u_t + u_{xxx} = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x,0) = f(x) & \text{on } \mathbb{R} \times \{t = 0\}, \\ \lim_{|x|\to\infty} u(x,t) = 0 & \text{for } \ t \in [0, \infty). \end{cases} \tag{2.15}$$

Solution. We apply Fourier transform with respect to the variable x:

$$\begin{cases} \widehat{u}_t - i\omega^3 \widehat{u} = 0, \\ \widehat{u}(\omega, 0) = \widehat{f}(\omega). \end{cases}$$

Then $\widehat{u}(\omega, t) = e^{i\omega^3 t}\widehat{f}(\omega)$, and u is recovered by taking the inverse Fourier transform of $\widehat{u}$. By convolution theorem, $u = G(\cdot, t) * f$, where $G(\cdot, t)$ is the inverse Fourier transform of $e^{i\omega^3 t}$ up to a factor $1/\sqrt{2\pi}$:

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega^3 t} e^{i\omega x}\, d\omega.$$

We compute the function G by constructing an ordinary differential equation for it. Fix $t = \frac{1}{3}$, and consider the function $g(x) = G(x, \frac{1}{3})$. Then

$$\widehat{g}(\omega) = \frac{1}{\sqrt{2\pi}} e^{i\omega^3/3}.$$

By Proposition 2.7,

$$g'' = \mathcal{F}^{-1}(-\omega^2 \widehat{g}) = -\frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left(\omega^2 e^{i\omega^3/3}\right), \quad \text{and} \quad xg = -i\mathcal{F}^{-1}(\widehat{g}') = \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}\left(\omega^2 e^{i\omega^3/3}\right).$$

Hence the function g satisfies the *Airy equation* $g'' - xg = 0$. Since our solution should vanish at infinity, we take the solution $g(x) = \mathrm{Ai}(x)$. For general $t > 0$, applying change of variable gives

$$G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{1}{3}i\left(\sqrt[3]{3t}\omega\right)^3} e^{i\omega x}\, d\omega = \frac{1}{\sqrt[3]{3t}} \mathrm{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right).$$

The solution to the problem (2.15) is $u(\cdot, t) = G(\cdot, t) * f$.

3 Distribution Theory

3.1 Topology on $C_c^\infty(U)$

The Fréchet space $\mathcal{D}(K)$. Let K be a compact set of $\mathbb{R}^n$. The space $C_c^\infty(K)$ is defined to be the set of C^∞ functions on $\mathbb{R}^n$ whose support is compact and contained in K. This space is a Fréchet space with the topology $\mathscr{T}_K$ defined by the norms

$$\|\phi\|_{K,N} = \sup_{x\in K, |\alpha|\le N} |\partial^\alpha \phi(x)|, \quad N \in \mathbb{N}_0.$$

That is, a local base for this topology at $\phi \in C_c^\infty(K)$ is the family of sets

$$U_{K,N}^\epsilon(\phi) = \left\{\psi \in C_c^\infty(K) : \|\psi - \phi\|_{K,N} < \epsilon\right\},$$

where $N \in \mathbb{N}_0$ and $\epsilon > 0$. Indeed, we only need to define the base sets

$$U_{K,N}^\epsilon = \left\{\psi \in C_c^\infty(K) : \|\psi\|_{K,N} < \epsilon\right\}, \quad N \in \mathbb{N}_0,\ \epsilon > 0$$

at 0, and take $\phi + U_{K,N}^\epsilon$ to be the base sets at ϕ. The Fréchet space $C_c^\infty(K)$ is metrizable by setting

$$d_K(\phi,\psi) = \sum_{N=1}^\infty \frac{1}{2^N}\frac{\|\phi-\psi\|_{K,N}}{1+\|\phi-\psi\|_{K,N}}, \quad \phi,\psi \in C_c^\infty(K).$$

We denote by $\mathcal{D}(K)$ the space $C_c^\infty(K)$ endowed with the topology $\mathscr{T}_K$. In $\mathcal{D}(K)$, every sequence (ϕ_k) converges to ϕ if and only if $\partial^\alpha\phi_k \to \partial^\alpha\phi$ uniformly for all multi-indices α.

Construct a base for a topology on $C_c^\infty(U)$. For an open set $U \subset \mathbb{R}^n$, the space $C_c^\infty(U)$ is defined to be the set of C^∞ functions whose support is compact and contained in U. Indeed, $C_c^\infty(U)$ can be viewed as the union of spaces $C_c^\infty(K)$ as K ranges over all compact subsets of U.

To construct a topology on $C_c^\infty(U)$, let $\mathscr{B}_0$ be the family of all balanced[1], convex sets $V \subset C_c^\infty(U)$ such that $V \cap C_c^\infty(K) \in \mathscr{T}_K$ for all compact $K \subset U$. We can show that $\mathscr{B}_0$ is nonempty. For example, let

$$V_N^\epsilon = \left\{\psi \in C_c^\infty(U) : \sup_{x\in U,|\alpha|\le N} |\partial^\alpha\psi(x)| < \epsilon\right\}. \tag{3.1}$$

Then V_N^ϵ is balanced, convex, and $V_N^\epsilon \cap C_c^\infty(K) = U_{K,N}^\epsilon \in \mathscr{T}_K$. We then define

$$\mathscr{B} = \left\{\phi + V : \phi \in C_c^\infty(U), V \in \mathscr{B}_0\right\}.$$

The sets in $\mathscr{B}$ gives an appropriate topology on $C_c^\infty(U)$.

Theorem 3.1. *The family $\mathscr{B}$ is a base for a locally convex Hausdorff topology $\mathscr{T}$ on $C_c^\infty(U)$ that turns $C_c^\infty(U)$ into a topological vector space.*

Remark. We write for $\mathcal{D}(U)$ the topological space $(C_c^\infty(U), \mathscr{T})$. Its elements are called *testing functions.*

Proof. **Step I.** We first verify that $\mathscr{B}$ is a base for a topology on $C_c^\infty(U)$. It suffices to verify the following two conditions:

(i) For each $\phi \in C_c^\infty(U)$ there exists $U \in \mathscr{B}$ such that $\phi \in U$;

(ii) For each $U_1, U_2 \in \mathscr{B}$ with $U_1 \cap U_2 \neq \emptyset$ and each $\phi \in U_1 \cap U_2$, there exists $V \in \mathscr{B}$ such that $V \ni \phi$ and $V \subset U_1 \cap U_2$. In other words, $\mathscr{B}$ is closed under finite intersection operation.

[1]A subset E of a vector space X is balanced if $tx \in E$ for all $x \in E$ and $|t| \le 1$.

• For (i), we let $\phi \in C_c^\infty(U)$, $N \in \mathbb{N}_0$ and $\epsilon > 0$. The set V_N^ϵ defined in (3.1) is in $\mathscr{B}_0$, and $\phi + V_N^\epsilon \in \mathscr{B}$.

• For (ii), we let $\phi_1, \phi_2 \in C_c^\infty(U)$ and $V_1, V_2 \in \mathscr{B}_0$ be such that $(\phi_1 + V_1) \cap (\phi_2 + V_2) \neq \emptyset$. We fix any $\phi \in (\phi_1 + V_1) \cap (\phi_2 + V_2)$, and take a compact set $K \subset U$ such that K contains the supports of ϕ_1, ϕ_2 and ϕ. Then for $j = 1, 2$, we have

$$\phi - \phi_j \in V_j \cap C_c^\infty(K) \in \mathscr{T}_K.$$

Using the continuity of scalar multiplication in $C_c^\infty(K)$, we may find $0 < \alpha < 1$, such that

$$\phi - \phi_j \in (1-\alpha)(V_j \cap C_c^\infty(K)) \subset (1-\alpha)V_j, \quad j = 1, 2.$$

By convexity of the sets V_j, we have

$$\phi - \phi_j + \alpha V_j = (1-\alpha)V_j + \alpha V_j = V_j, \quad j = 1, 2,$$

so that $\phi + \alpha V_j \in \phi_j + V_j$ for $j = 1, 2$, and $\phi + \alpha(V_1 \cap V_2) \subset (\phi_1 + V_1) \cap (\phi_2 + V_2)$. Hence $\mathscr{B}$ is a base for a topology $\mathscr{T}$ given by all unions of members of $\mathscr{B}$.

Step II. Next we verify that $C_c^\infty(U)$ is a topological vector space under $\mathscr{T}$.

• To prove the continuity of scalar multiplication at a point $(t_0, \phi_0) \in \mathbb{C} \times C_c^\infty(U)$, we notice that each neighborhood of $t_0\phi_0$ contains some $t_0\phi_0 + V$, where $V \in \mathscr{B}_0$. Let $K = \operatorname{supp}(\phi_0)$. Then $\phi_0 \in \mathcal{D}(K)$. By continuity of scalar multiplication in $\mathcal{D}(K)$, we may find $\gamma > 0$ so small that

$$\gamma\phi_0 \in \frac{1}{2}\left(V \cap C_c^\infty(K)\right) \subset \frac{1}{2}V.$$

Let $s = \frac{1}{2(|t_0|+\gamma)}$. Then for every $|t - t_0| < \gamma$ and $\phi \in \phi_0 + sV$,

$$t\phi - t_0\phi_0 = t(\phi - \phi_0) + (t - t_0)\phi \in tsV + \frac{1}{2}V \subset \frac{1}{2}V + \frac{1}{2}V = V,$$

where we use the fact that V is convex and balanced. Therefore $t\phi \in t_0\phi_0 + V$ for every $|t - t_0| < \gamma$ and $\phi \in \phi_0 + sV$, which proves the continuity of scalar multiplication.

• To prove the continuity of addition at a point $(\phi_1, \phi_2) \in C_c^\infty(U) \times C_c^\infty(U)$, consider a neighborhood $\phi_1 + \phi_2 + V$ of $\phi_1 + \phi_2$, where $V \in \mathscr{B}_0$. The convexity of V implies that

$$\left(\phi_1 + \frac{1}{2}V\right) + \left(\phi_2 + \frac{1}{2}V\right) = \phi_1 + \phi_2 + V.$$

Since $V \cap \mathcal{D}(K) \in \mathscr{T}_K$ for all compact $K \subset U$, and since the scalar multiplication is continuous in $\mathcal{D}(K)$, we have $\frac{1}{2}V \cap \mathcal{D}(K) \in \mathscr{T}_K$ for all compact $K \subset U$, and $\frac{1}{2}V \in \mathscr{B}_0$. Hence both $\phi_1 + \frac{1}{2}V$ and $\phi_2 + \frac{1}{2}V$ are in $\mathscr{B}$, and the addition operation is continuous.

Step III. Finally, to prove that $(C_c^\infty(U), \mathscr{T})$ is a Hausdorff space, we take $\phi_1 \neq \phi_2$ from $C_c^\infty(U)$ and define

$$V = \left\{\psi \in C_c^\infty(U) : \sup_{x \in U} |\psi(x)| < \frac{1}{2} \sup_{x \in U} |\phi_1(x) - \phi_2(x)|\right\}.$$

In view of (3.1), we have $V \in \mathscr{B}_0$. If $\phi \in (\phi_1 + V) \cap (\phi_2 + V)$, we have

$$\begin{aligned}\sup_{x \in U} |\phi_1(x) - \phi_2(x)| &\leq \sup_{x \in U} |\phi(x) - \phi_1(x)| + \sup_{x \in U} |\phi(x) - \phi_2(x)| \\ &< \frac{1}{2} \sup_{x \in U} |\phi_1(x) - \phi_2(x)| + \frac{1}{2} \sup_{x \in U} |\phi_1(x) - \phi_2(x)| = \sup_{x \in U} |\phi_1(x) - \phi_2(x)|,\end{aligned}$$

a contradiction! Hence $(\phi_1 + V) \cap (\phi_2 + V) = \emptyset$, and we finish the proof. □

We now show that the topology $\mathscr{T}$, when restricted to $\mathcal{D}(K)$, for some compact set $K \subset U$, does not produce more open sets than the ones in $\mathscr{T}_K$.

Proposition 3.2. *Let $U \subset \mathbb{R}^n$ be an open set. For every compact set $K \subset U$, the topology on $\mathcal{D}(K)$ coincide with the relative topology of $\mathcal{D}(K)$ as a subspace of $\mathcal{D}(U)$.*

Proof. Fix a compact set $K \subset U$ and let $W \in \mathscr{T}$. We claim $W \cap \mathcal{D}(K) \in \mathscr{T}_K$. We may assume $W \cap \mathcal{D}(K)$ is nonempty, otherwise the claim is clear. Let $\phi \in W \cap \mathcal{D}(K)$. Since $\mathscr{B}$ is a base for $\mathscr{T}$, we take $V \in \mathscr{B}_0$ such that $\phi + V \subset W$. Then $\phi + (V \cap \mathcal{D}(K)) \subset W \cap \mathcal{D}_K$, and $\phi + (V \cap \mathcal{D}(K)) \in \mathscr{T}_K$ since $\phi \in \mathcal{D}(K)$ and $V \cap \mathcal{D}(K) \in \mathscr{T}_K$. Hence every point of $W \cap \mathcal{D}(K)$ is in the interior with respect to $\mathscr{T}_K$, and $W \cap \mathcal{D}(K) \in \mathscr{T}_K$.

Conversely, let $W \subset \mathscr{T}_K$. We claim that $W = V \cap \mathcal{D}(K)$ for some open $V \in \mathscr{T}$. Since the family of sets $U_{K,N}^{\epsilon}$ is a local base for the topology $\mathscr{T}_K$, for each $\phi \in W$, we may find $N_\phi \in \mathbb{N}_0$ and $\epsilon_\phi > 0$ such that $\phi + U_{K,N_\phi}^{\epsilon_\phi} \subset W$. Let $V_{N_\phi}^{\epsilon_\phi}$ be defined as in (3.1). Then

$$(\phi + V_{N_\phi}^{\epsilon_\phi}) \cap \mathcal{D}(K) = \phi + U_{K,N_\phi}^{\epsilon_\phi} \subset W,$$

and $\phi + V_{N_\phi}^{\epsilon_\phi} \in \mathscr{B}$. Therefore $V = \bigcup_{\phi \in W} (\phi + V_{N_\phi}^{\epsilon_\phi})$ is a set in $\mathscr{T}$ with the desired property. □

Proposition 3.3. *Let $U \subset \mathbb{R}^n$ be an open set. If $W \subset \mathcal{D}(U)$ is topologically bounded, there exists a compact set $K \subset U$ such that $W \subset \mathcal{D}(K)$.*

Proof. Assume that W is not contained in $\mathcal{D}(K)$ for any compact $K \subset U$. We take an increasing sequence (K_j) of compact sets such that $K_j \subset \mathring{K}_{j+1}$ for all $j \in \mathbb{N}$ and $U = \bigcup_{j=1}^{\infty} K_j$. Then we may find for each $j \in \mathbb{N}$ a function $\phi_j \in W$ and a point $x_j \in K_{j+1} \backslash K_j$ such that $\phi_j(x_j) \neq 0$. Define

$$V = \left\{ \phi \in \mathcal{D}(U) : |\phi(x_j)| < \frac{1}{j} |\phi_j(x_j)| \text{ for all } j \in \mathbb{N} \right\}.$$

Since each compact set $K \subset U$ contains only finitely many x_j, we have $V \cap \mathcal{D}(K) \in \mathscr{T}_K$, and so $V \subset \mathscr{T}$. Since W is topologically bounded, there exists $t > 0$ such that $W \subset tV$. If an integer $N \geq t$, we have $\phi_N(x_N) \neq 0$, and $t^{-1}|\phi_N(x_N)| \geq N^{-1}|\phi_N(x_N)|$. Hence $t^{-1}\phi_N \notin V$, and $\phi_N \notin tV$. However $\phi_N \in W \subset tV$, which yields a contradiction. Hence there exists a compact $K \subset U$ with $\mathcal{D}(K) \supset W$. □

The topology on $\mathcal{D}(U)$ is complete, and convergent sequence in $\mathcal{D}(U)$ can be explicitly characterized.

Proposition 3.4. *Let $U \subset \mathbb{R}^n$. The space $\mathcal{D}(U)$ is complete. Furthermore, a sequence (ϕ_j) in $\mathcal{D}(U)$ converges to $\phi \in \mathcal{D}(U)$ if and only if*

(i) there exists a compact set $K \subset U$ such that $(\phi_j) \subset \mathcal{D}(K)$, and

(ii) $\lim_{j \to \infty} \partial^\alpha \phi_j = \partial^\alpha \phi$ uniformly on K for each multi-index $\alpha \in \mathbb{N}_0^n$.

Proof. Let (ϕ_j) be a Cauchy sequence in $\mathcal{D}(U)$. Then (ϕ_j) is topologically bounded, and by Proposition 3.3, there exists a compact set $K \subset U$ such that $(\phi_j) \subset \mathcal{D}(K)$. By Proposition 3.2, we obtain a Cauchy sequence (ϕ_j) in $\mathcal{D}(K)$. Therefore, for every $N \in \mathbb{N}_0$ and every $\epsilon > 0$, there exists M such that

$$\sup_{x \in K, |\alpha| \leq N} |\phi_j(x) - \phi_k(x)| < \epsilon$$

for all $j, k \geq M$. Consequently, for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$, the Cauchy sequence $\{\partial^\alpha \phi_j\}$ converges uniformly in K to a continuous function $\psi_\alpha \in C_c(K)$. An inductive argument using the fundamental theorem of calculus shows that $\partial^\alpha \psi_0 = \psi_\alpha$ for every multi-index $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq N$. Given the arbitrariness of $N \in \mathbb{N}_0$, we conclude that $\psi_0 \in \mathcal{D}(K)$ and that the sequence (ϕ_j) converges to ψ_0 with respect to $\mathscr{T}$. Hence the space $\mathcal{D}(U)$ is complete.

Conversely, if a sequence (ϕ_j) in $\mathcal{D}(U)$ satisfies conditions (i) and (ii), it converges to ϕ in $\mathcal{D}(K)$. By Proposition 3.3, it also converges to ϕ in $\mathcal{D}(U)$. □

Now we discuss the continuous mappings on $\mathcal{D}(U)$.

Proposition 3.5. *Let $U \subset \mathbb{R}^n$ be an open set, X a locally convex topological vector space, and $T : \mathcal{D}(U) \to X$ a linear operator. The following properties are equivalent:*

(i) T is continuous.

(ii) T is bounded, i.e. it sends topologically bounded sets of $\mathcal{D}(U)$ into topologically bounded sets of X.

(iii) If (ϕ_j) converges to ϕ in $\mathcal{D}(U)$, then $\lim_{j\to\infty} T\phi_j = T\phi$.

(iv) The restriction of T to $\mathcal{D}(K)$ is continuous for every compact set $K \subset U$.

If $X = \mathbb{C}$, the following statement is also equivalent to above all:

(v) For every compact set $K \subset U$, there exists an integer $N \in \mathbb{N}_0$ and a constant $c_K > 0$ such that $|T\phi| \leq c_K \|\phi\|_{K,N}$ for all $\phi \in \mathcal{D}(K)$.

Proof. We prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

• (i) $\Rightarrow$ (ii). Suppose that $T : \mathcal{D} \to X$ is continuous, and $W \subset \mathcal{D}(U)$ is a topologically bounded set. If V is a neighborhood of 0 in X, then $T^{-1}(V)$ is a neighborhood of 0 in $\mathcal{D}(X)$, and there exists $t > 0$ such that $W \subset tT^{-1}(V)$. Consequently $T(W) \subset tV$. Hence $T(W)$ is also topologically bounded.

• (ii) $\Rightarrow$ (iii). We may assume $(\phi_j) \to 0$ by replacing (ϕ_j) with $(\phi_j - \phi)$. By Proposition 3.4, there exists a compact set K such that $(\phi_j) \subset \mathcal{D}(K)$, and $d_K(\phi_j, 0) \to 0$ as $j \to \infty$.

Let $B = \{\phi \in \mathcal{D}(K) : d_K(\phi, 0) < 1\}$ be the unit ball in $\mathcal{D}(K)$ centered at 0. If T is bounded, the set $T(B)$ is topologically bounded. Then for any neighborhood V of 0 in X, there exists $t > 0$ such that $T(B) \subset tV$, so $T(t^{-1}B) \subset V$. Since $d_K(\phi_j, 0) \to 0$ as $j \to 0$, there exists N such that $\phi_j \in t^{-1}(B)$ for all $j \geq N$. Hence $(T\phi_j)$ is eventually in V, and $T\phi_j$ converges to 0.

• (iii) $\Rightarrow$ (iv). Fix a compact set $K \subset U$. If (ϕ_j) is a sequence in $\mathcal{D}(K)$ such that $d_K(\phi_j, 0) \to 0$ as $j \to \infty$, by Proposition 3.4, we have $\phi_j \to 0$ in $\mathcal{D}(U)$, and $T\phi = \lim_{j\to\infty} T\phi_j$ by property (iii). Hence the restriction of T to $\mathcal{D}(K)$ is continuous at 0. By linearity, the restriction is continuous.

• (iv) $\Rightarrow$ (i). For every neighborhood V of 0 in X and every compact set $K \subset U$, the restriction of T to $\mathcal{D}(K)$ is continuous at zero, and $T^{-1}(V) \cap \mathcal{D}(K) \in \mathscr{T}_K$. Since K is arbitrary, $T^{-1}(V) \in \mathscr{T}$. Therefore, T is continuous at 0 and, by linearity, everywhere in $\mathcal{D}(U)$.

• (iv) $\Leftrightarrow$ (v). Let $X = \mathbb{C}$. Assume that (iv) holds and fix a compact $K \subset U$. By continuity of $T|_{\mathcal{D}(K)}$ at the origin, there exists $N \in \mathbb{N}_0$ and $\epsilon > 0$ such that $U^{\epsilon}_{K,N} \subset T^{-1}(\{|z| < 1\})$, that is, $|T\phi| < 1$ for all $\phi \in \mathcal{D}(K)$ with $\|\phi\|_{K,N} < \epsilon$. If $\phi \in \mathcal{D}(K)$ and $\phi \neq 0$, then $\|\phi\|_{K,N} \neq 0$, and by linearity of T, we have $|T\phi| \leq \frac{2}{\epsilon}\|\phi\|_{K,N}$. Conversely, if (v) holds, for any $\delta > 0$, by taking $\epsilon > 0$ sufficiently small, we have $|T\phi| < \delta$ for all $\phi \in U^{\epsilon}_{K,N}$. Hence the restriction $T|_{\mathcal{D}(K)}$ is continuous. ☐

Proposition 3.6. *Let U and U' be open subsets of $\mathbb{R}^n$, and $T : \mathcal{D}(U) \to \mathcal{D}(U')$ a linear operator. The following properties are equivalent:*

(i) T is continuous if and only if

(ii) for each compact set $K \subset U$, there exists a compact set $K' \subset U'$ such that $T(\mathcal{D}(K)) \subset \mathcal{D}(K')$, and the restriction $T : \mathcal{D}(K) \to \mathcal{D}(K')$ is continuous.

Proof. (ii) $\Rightarrow$ (i) is a special case of the implication (iv) $\Rightarrow$ (i) in Proposition 3.5. To prove (i) $\Rightarrow$ (ii), we let $T : \mathcal{D}(U) \to \mathcal{D}(U')$ be a continuous linear operator and fix a compact set $K \subset U$. According to the implication (i) $\Rightarrow$ (iv) in Proposition 3.5, the restriction of T to $\mathcal{D}(K)$ is continuous. If we can show that $T(\mathcal{D}(K)) \subset \mathcal{D}(K')$ for some compact $K' \subset U'$, the proof will be completed by Proposition 3.2.

Assume that $T(\mathcal{D}(K))$ is not contained in $\mathcal{D}(K')$ for any compact $K' \subset U'$. Take an increasing sequence (K'_j) of compact sets such that $K'_j \subset \mathring{K}'_{j+1}$ for all $j \in \mathbb{N}$ and $U' = \bigcup_{j=1}^{\infty} K'_j$. Then we may find for each $j \in \mathbb{N}$ a function $\phi_j \in \mathcal{D}(U')$ and a point $x_j \in K'_{j+1} \backslash K'_j$ such that $d_K(\phi_j, 0) = 1$ and $(T\phi_j)(x_j) \neq 0$. Since (ϕ_j) is topologically bounded in $\mathcal{D}(U)$, by Proposition 3.5 (ii), $(T\phi_j)$ is topologically bounded in $\mathcal{D}(U')$, and by Proposition 3.3, there exists $K' \subset U'$ such that $(\phi_j) \subset \mathcal{D}(K')$, which is contradiction! ☐

3.2 Distributions

Motivation. Let $f \in L^p(\mathbb{R}^n)$, where $1 < p \le \infty$. For $q = \frac{p}{p-1}$, we define $T_f : L^q(\mathbb{R}^n) \to \mathbb{C}$ by

$$T_f g = \int_{\mathbb{R}^n} f(x)g(x)\,dx, \quad g \in L^q(\mathbb{R}^n).$$

The Riesz representation theorem states that the map $f \mapsto T_f$ is an isometric isomorphism of $L^p(\mathbb{R}^n)$ onto the dual space $L^q(\mathbb{R}^n)^*$ of $L^q(\mathbb{R}^n)$. In other words, $f \in L^p(\mathbb{R}^n)$ is completely determined by its action as a bounded linear functional on $L^q(\mathbb{R}^n)$. On the other hand, by Lebesgue differentiation theorem,

$$\lim_{r\to 0^+} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\,dy = f(x), \quad \text{for } a.e.\ x \in \mathbb{R}^n,$$

where $B(x,r)$ is the (open) ball of radius r about x, and m is the Lebesgue measure. Hence if we take $g = m(B(x,r))^{-1}\chi_{B(x,r)}$, we can recover the pointwise value of f for almost every $x \in \mathbb{R}^n$ as $r \to 0$. Thus, we lose nothing by thinking of f as a linear mapping from $L^q(\mathbb{R}^n)$ to $\mathbb{C}$ rather than a map from $\mathbb{R}^n$ to $\mathbb{C}$.

The idea of distribution follows by allowing $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and requiring $\phi \in \mathcal{D}(\mathbb{R}^n)$. The map T_f defines a linear functional on $\mathcal{D}(\mathbb{R}^n)$, and the pointwise values of f can be recovered a.e. by a similar approach of Theorem 1.9. Nevertheless, there are also linear functionals on $\mathcal{D}(\mathbb{R}^n)$ that are not of the form T_f.

Definition 3.7 (Distribution)**.** Let U be an open subset of $\mathbb{R}^n$. A *distribution* on U is a continuous linear functional on $\mathcal{D}(U)$. The space of all distributions on U is denoted by $\mathcal{D}'(U)$. We equip $\mathcal{D}'(U)$ with the weak* topology, i.e. the neighborhoods of $T_0 \in \mathcal{D}'(U)$ is generated by the sets

$$U^{\epsilon}_{f_1,\cdots,f_m}(T_0) = \{T \in \mathcal{D}'(U) : |Tf_j - T_0 f_j| < \epsilon,\ j = 1, 2, \cdots, m\},$$

where $\epsilon > 0$, $m \in \mathbb{N}$ and $f_1, \cdots, f_m \in C_c^\infty(U)$. Furthermore, a sequence $T_j \to T$ in the weak* topology if and only if $T_j f \to Tf$ for all $f \in C_c^\infty(U)$.

Notations. If $F \in \mathcal{D}'(U)$ and $\phi \in C_c^\infty(U)$, we use the pairing notation $\langle F, \phi\rangle$ for the value of F evaluated at the point ϕ. Sometimes it is helpful to pretend that a distribution $F \in \mathcal{D}'(U)$ is a function on U even when it really is not, and to write $\int_U F(x)\phi(x)\,dx$ instead of $\langle F, \phi\rangle$.

We shall use a tilde to denote the reflection of a function in the origin: $\widetilde{\phi}(x) = \phi(-x)$.

Example 3.8. Following are some examples of distribution on an open set $U \subset \mathbb{R}^n$:

- Every function $f \in L^1_{\text{loc}}(U)$ defines a distribution on U, namely, the functional $\phi \mapsto \int f\phi\,dx$. Clearly, two functions that are equal a.e. define the same distribution, since they are identified in $L^1_{\text{loc}}(U)$.
- Every Radon measure μ on U defines a distribution $\phi \mapsto \int \phi\,d\mu$.
- For a point $x_0 \in U$ and a multi-index $\alpha \in \mathbb{N}_0^n$, the map $\phi \mapsto \partial^\alpha\phi(x_0)$ defines a distribution that does not arise from a function.
- In particular, when $U = \mathbb{R}^n$, $\alpha = 0$ and $x = 0$, this distribution arise from a measure μ which is the point mass at the origin 0. We call this distribution the *Dirac δ-function*, denoted by δ:

$$\langle \delta, \phi\rangle = \phi(0), \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

It can be represented heuristically as

$$\delta(x) = \begin{cases} \infty, & x = 0, \\ 0, & x \neq 0, \end{cases}$$

and we write $\int_{\mathbb{R}^n} \delta(x)\phi(x)\,dx = \phi(0)$.

We have the following approximation for Dirac δ-function.

Proposition 3.9. *Assume $f \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f(x)\,dx = 1$. Define*

$$f_t(x) = \frac{1}{t^n} f\left(\frac{x}{t}\right), \quad t > 0.$$

Then $f_t \to \delta$ in $\mathcal{D}'(\mathbb{R}^n)$ as $t \to 0$.

Proof. If $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle f_t, \phi \rangle = \int_{\mathbb{R}^n} f_t(x)\phi(x)\,dx = \int_{\mathbb{R}^n} f_t(x)\widetilde{\phi}(-x)\,dx = (f_t * \widetilde{\phi})(0),$$

which converges to $\widetilde{\phi}(0) = \phi(0) = \langle \delta, \phi \rangle$ as $t \to 0$ by Proposition 1.6. $\square$

Let $F \in \mathcal{D}'(U)$ be a distribution on an open set $U \subset \mathbb{R}^n$. For an open set $V \subset U$, we say $F = 0$ on V if $\langle F, \phi \rangle = 0$ for all $\phi \in C_c^\infty(V)$ (for example, if $F \in L^1_{\text{loc}}(U)$, it means that $F = 0$ a.e. on V). Since a function in $C_c^\infty(V_1 \cup V_2)$ need not to be supported in either V_1 or V_2, it is not so clear that $F = 0$ on both V_1 and V_2 implies $F = 0$ on $V_1 \cup V_2$. Nevertheless, it is true:

Proposition 3.10. *Let $(V_\alpha)_{\alpha \in A}$ be a collection of open subsets of U, and $V = \bigcup_{\alpha \in A} V_\alpha$. If $F \in \mathcal{D}'(U)$ and $F = 0$ on each V_α, then $F = 0$ on V.*

Proof. If $\phi \in C_c^\infty(V)$, by compactness, there exist finitely many $\alpha_1, \cdots, \alpha_m \in A$ such that $\text{supp}(\phi) \subset \bigcup_{j=1}^m V_{\alpha_j}$. Take a smooth partition of unity $(\psi_j)_{j=1}^m$, i.e. $\text{supp}(\psi_j) \subset V_{\alpha_j}$ for each j and $\sum_{j=1}^m \psi_j = 1$ on $\text{supp}(\phi)$. Then

$$\langle F, \phi \rangle = \sum_{j=1}^m \langle F, \phi\psi_j \rangle = 0.$$

Hence $F = 0$ on V. $\square$

Remark I. According to this proposition, we can take a maximal open set W on which $F = 0$, namely the union of all open sets on which $F = 0$. Its complement $U \backslash W$ is called the *support of F*.

Remark II. More generally, we say two distributions $F, G \in \mathcal{D}'(V)$ *agree* on an open set $V \subset U$ if $F - G = 0$ on V. According to this proposition, if two distributions agree on each member of a collection of open sets, they also agree on the union of those open sets.

Operations on distributions. Let $U \subset \mathbb{R}^n$ be an open set, and $F \in \mathcal{D}'(U)$.

(i) (Product). If $\psi \in C^\infty(U)$, we define the *product* ψF to be

$$\langle \psi F, \phi \rangle = \langle F, \psi\phi \rangle, \quad \phi \in \mathcal{D}(U).$$

For any compact $K \subset U$ and any sequence $\phi_j \in C_c^\infty(K)$ that converges to ϕ in $\mathcal{D}(K)$, since $\psi\phi_j \to \psi\phi$ and $F|_{\mathcal{D}(K)}$ is continuous, we have $\langle F, \psi\phi_j \rangle \to \langle F, \psi\phi \rangle$. Hence $\psi F \in \mathcal{D}'(U)$.

(ii) (Translation). If $y \in \mathbb{R}^n$ and $F \in L^1_{\text{loc}}(U)$,

$$\int_{U+y} F(x-y)\phi(x)\,dx = \int_U F(x)\phi(x+y)\,dx, \quad \phi \in \mathcal{D}(U+y).$$

Similarly, for $F \in \mathcal{D}'(U)$, we define the *translated distribution* $\tau_y F$ to be

$$\langle \tau_y F, \phi \rangle = \langle F, \tau_{-y}\phi \rangle, \quad \phi \in \mathcal{D}(U+y).$$

Then $\tau_y F \in \mathcal{D}'(U+y)$. In particular, the point mass at y is $\tau_y \delta$.

(iii) (Composition with linear map). If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation and $F \in L^1_{\text{loc}}(U)$,

$$\int_U F(Tx)\phi(x)\,dx = |\det(T)|^{-1} \int_{T^{-1}(U)} F(y)\phi(T^{-1}y)\,dy, \quad \phi \in \mathcal{D}(T^{-1}(U)).$$

Similarly, for $F \in \mathcal{D}'(U)$, we define the *composition* $F \circ T$ to be

$$\langle F \circ T, \phi \rangle = |\det(T)|^{-1} \langle F, \phi \circ T^{-1} \rangle, \quad \phi \in \mathcal{D}(T^{-1}(U)).$$

Then $F \circ T = \mathcal{D}'(T^{-1}(U))$. In particular, if $Tx = -x$, we define the *reflection of* F *in the origin* by

$$\langle \widetilde{F}, \phi \rangle = \langle F, \widetilde{\phi} \rangle, \quad \phi \in \mathcal{D}^\infty(-U).$$

(iv) (Convolution). Given $\psi \in C_c^\infty(\mathbb{R}^n)$, let $V = \{x : x - y \in U \text{ for all } y \in \operatorname{supp}(\psi)\}$. If $F \in L^1_{\text{loc}}(U)$,

$$(F * \psi)(x) = \int_U F(y)\psi(x-y)\,dy = \int_U F(y)(\tau_x \widetilde{\psi})(y)\,dy, \quad x \in V,$$

and by Fubini's theorem, if $\phi \in C_c^\infty(V)$,

$$\begin{aligned}\int_V (F*\psi)(x)\phi(x)\,dx &= \int_V \int_U F(y)\psi(x-y)\phi(x)\,dy\,dx \\ &= \int_U \int_V F(y)\widetilde{\psi}(y-x)\phi(x)\,dx\,dy = \int_U F(y)(\phi * \widetilde{\psi})(y)\,dy.\end{aligned}$$

For $F \in \mathcal{D}'(U)$, we have two approaches to define the *convolution* $F * \psi$:

– Analogous to the first identity, define $F * \psi$ be the function

$$(F * \psi)(x) = \langle F, \tau_x \widetilde{\psi} \rangle, \quad x \in V.$$

– Analogous to the second identity, define $F * \psi$ be the mapping

$$\langle F * \psi, \phi \rangle = \langle F, \phi * \widetilde{\psi} \rangle, \quad \phi \in \mathcal{D}(V).$$

If $K \subset V$ is compact and $(\phi_j) \subset C_c^\infty(K)$ is a sequence converging to ϕ in $\mathcal{D}(K)$, we have

$$\partial^\alpha(\phi_j * \widetilde{\psi}) = (\partial^\alpha \phi_j) * \widetilde{\psi} \to (\partial^\alpha \phi) * \widetilde{\psi} = \partial^\alpha(\phi * \widetilde{\psi})$$

uniformly for all multi-indices $\alpha \in \mathbb{N}_0^n$. Hence $(F * \psi)|_{\mathcal{D}(K)}$ is continuous, and $F * \psi \in \mathcal{D}'(V)$.

The following proposition shows that the two definitions of the convolution $F * \psi$ coincide. Furthermore, the distribution as a function on U is infinitely differentiable.

Proposition 3.11. *Let $U \subset \mathbb{R}^n$ be open. Given $\psi \in C_c^\infty(\mathbb{R}^n)$, let $V = \{x : x - y \in U$ for all $y \in \operatorname{supp}(\psi)\}$. For $F \in \mathcal{D}'(U)$, define $(F * \psi)(x) = \langle F, \tau_x \widetilde{\psi} \rangle$ for all $x \in V$. Then*

(i) $F * \psi \in C^\infty(V)$, and $\partial^\alpha(F * \psi) = F * (\partial^\alpha \psi)$ for all multi-indices $\alpha \in \mathbb{N}_0^n$;

(ii) For all $\phi \in C_c^\infty(V)$, we have $\int_V (F * \psi)(x)\phi(x)\,dx = \langle F, \phi * \widetilde{\psi} \rangle$.

Proof. If $x \in V$, by Proposition 1.5, we have $\tau_{x+s}\widetilde{\psi} \to \tau_x \widetilde{\psi}$ uniformly as $s \to 0$, and the same holds for all partial derivatives. Then $\tau_{x+s}\widetilde{\psi} \to \tau_x \widetilde{\psi}$ in $\mathcal{D}(U)$ as $s \to 0$. By continuity of F on $\mathcal{D}(U)$ we have that $\langle F, \tau_x \widetilde{\psi} \rangle$ is continuous in x. Furthermore, for any $j = 1, 2, \cdots, n$, we have

$$\left| \frac{\psi(x + he_j - y) - \psi(x-y)}{h} - \partial_j \psi(x-y) \right| \le \sup_{t \in \mathbb{R}: |t| < |h|} |\partial_j \psi(x + te_j - y) - \partial_j \psi(x-y)|.$$

For any $\epsilon > 0$, by uniform continuity of $\partial_j\psi$, there exists a constant $\eta > 0$ independent of x and y such that the last bound is less than ϵ whenever $|h| < \eta$. Hence the difference quotient

$$\frac{\tau_{x+he_j}\widetilde{\psi} - \tau_x\widetilde{\psi}}{h} \to \tau_x\widetilde{\partial_j\psi} \tag{3.2}$$

uniformly as $h \to 0$. Since the same conclusion of difference quotient holds for all partial derivatives, the convergence (3.2) also holds in $\mathcal{D}(U)$. Therefore

$$\partial_j(F * \psi)(x) = \lim_{h\to 0} \frac{\langle F, \tau_{x+he_j}\widetilde{\psi}\rangle - \langle F, \tau_x\widetilde{\psi}\rangle}{h} = \langle F, \tau_x\widetilde{\partial_j\psi}\rangle = (F * \partial_j\psi)(x).$$

By induction on $|\alpha|$, we have $F * \psi \in C^\infty(V)$, and $\partial^\alpha(F * \psi) = F * \partial^\alpha\psi$. To prove the second result, we note that $\psi, \phi \in C_c^\infty(\mathbb{R}^n)$. Then we approximate the convolution $\phi * \widetilde{\psi}$ by Riemann sums:

$$(\phi * \widetilde{\psi})(x) = \int_{\mathbb{R}^n} \widetilde{\psi}(x-y)\phi(y)\, dy = \lim_{\epsilon\to 0^+} S_\epsilon(x) := \lim_{\epsilon\to 0^+} \epsilon^n \sum_{\kappa\in\mathbb{Z}^n} \widetilde{\psi}(x-\epsilon\kappa)\phi(\epsilon\kappa),$$

where there are finitely many nonzero terms when κ runs over $\mathbb{Z}^n$. The Riemann sums S_ϵ are supported in a common compact subset of U, and converges to $\phi * \widetilde{\psi}$ uniformly as $\epsilon \to 0$. Also, for all multi-indices α, $\partial^\alpha S_\epsilon = \epsilon^n \sum_{\kappa\in\mathbb{Z}^n} \partial^\alpha\widetilde{\psi}(x-\epsilon\kappa)\phi(\epsilon\kappa)$ converges uniformly to $\partial^\alpha(\phi * \widetilde{\psi})$. Hence $S_\epsilon \to \phi * \widetilde{\psi}$ in $\mathcal{D}(U)$, and

$$\langle F, \phi * \widetilde{\psi}\rangle = \lim_{\epsilon\to 0^+} \langle F, S_\epsilon\rangle = \lim_{\epsilon\to 0^+} \epsilon^n \sum_{\kappa\in\mathbb{Z}^n} \phi(\epsilon\kappa)\langle F, \tau_{\epsilon\kappa}\widetilde{\psi}\rangle = \int_V \phi(x)\langle F, \tau_x\widetilde{\psi}\rangle\, dx = \int_V (F * \psi)(x)\phi(x)\, dx.$$

Hence the two definitions of $F * \psi$ are equivalent. □

Next we show that although distributions can be highly singular objects, they can always be approximated by compactly supported smooth functions in the weak* topology.

Theorem 3.12. *For any open set $U \subset \mathbb{R}^n$, the space $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$ in the weak* topology.*

To prove this theorem we need some technical lemma.

Lemma 3.13. *Assume that $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$. Let $\psi_t(x) = t^{-n}\psi(t^{-1}x)$ for $t > 0$.*

(i) Given any neighborhood U of $\operatorname{supp}(\phi)$, we have $\operatorname{supp}(\phi * \psi_t) \subset U$ for $t > 0$ sufficiently small.

(ii) $\phi * \psi_t \to 0$ in $\mathcal{D}(\mathbb{R}^n)$ as $t \to 0$.

Proof. If $\operatorname{supp}(\psi) \subset \{x \in \mathbb{R}^n : |x| < R\}$, then $\operatorname{supp}(\phi * \psi_t)$ is contained in the set

$$V = \{x \in \mathbb{R}^n : d(x, \operatorname{supp}(\phi)) < tR\}.$$

When $t < R^{-1}d(\operatorname{supp}(\phi), U^c)$, the support of $\phi * \psi_t$ is contained in U. Moreover, by Propositions 1.3 and 1.6, $\partial^\alpha(\phi * \psi_t) = (\partial^\alpha\phi) * \psi_t \to \partial^\alpha t$ uniformly as $t \to 0$, and the second result follows. □

Proof of Theorem 3.12. Assume $F \in \mathcal{D}'(U)$. We first approximate F by distributions supported on compact subsets of U, then approximate the latter by functions in $C_c^\infty(U)$.

• Let (V_j) be a sequence of precompact open subsets of U increasing to U. For each j, by C^∞-Urysohn lemma [Proposition 1.10], we take $\zeta_j \in C_c^\infty(U)$ such that $\zeta_j = 1$ on $\overline{V}_j$. Given $\phi \in C_c^\infty(U)$, for j sufficiently large we have $\operatorname{supp}(\phi) \subset V_j$, and $\langle F, \phi\rangle = \langle F, \zeta_j\phi\rangle = \langle \zeta_j F, \phi\rangle$. Hence $\zeta_j F \to F$ in the weak* topology as $j \to \infty$.

• Let ψ and (ψ_t) be defined as in Lemma 3.13. Then $\phi * \widetilde{\psi}_t \to \phi$ in $\mathcal{D}(\mathbb{R}^n)$ as $t \to 0$. On the other hand, by Proposition 3.11, we have $(\zeta_j F) * \psi_t \in C^\infty(\mathbb{R}^n)$ and $\langle (\zeta_j F) * \psi_t, \phi\rangle = \langle \zeta_j F, \phi * \widetilde{\psi}_t\rangle \to \langle \zeta_j F, \phi\rangle$ as $t \to 0$. Hence $(\zeta_j F) * \psi_t \to \zeta_j F$ in $\mathcal{D}'(\mathbb{R}^n)$. Observing that $\operatorname{supp}(\zeta_j) \subset V_k$ for some k, if $\operatorname{supp}(\phi) \cap \overline{V}_k = \emptyset$, we have $\operatorname{supp}(\phi * \widetilde{\psi}_t) \cap \overline{V}_k = \emptyset$ for $t > 0$ sufficiently small, by Lemma 3.13, and $\langle (\zeta_j F) * \psi_t, \phi\rangle = \langle F, \zeta_j(\phi * \widetilde{\psi}_t)\rangle = 0$. Hence $\operatorname{supp}((\zeta_j F) * \psi_t) \subset \overline{V}_k \subset U$, and $(\zeta_j F) * \psi_t \in C_c^\infty(U)$ for j large enough and t small enough. □

Derivatives of distributions. Let U be an open subset of $\mathbb{R}^n$. If $f \in C_c^\infty(U)$, for any multi-index $\alpha \in \mathbb{N}_0^n$,

$$\int_U (\partial^\alpha f)(x)\phi(x)\,dx = (-1)^{|\alpha|}\int_U f(x)(\partial^\alpha \phi)(x)\,dx, \quad \phi \in C_c^\infty(U).$$

This is the integration by parts formula, where the boundary term vanishes since f is compactly supported. Generally, for $F \in \mathcal{D}'(U)$, we can define a linear functional $\partial^\alpha F$ on $C_c^\infty(U)$ by

$$\langle \partial^\alpha F, \phi\rangle = (-1)^{|\alpha|}\langle F, \partial^\alpha \phi\rangle, \quad \phi \in C_c^\infty(U).$$

For any compact $K \subset U$ and any sequence $(\phi_j) \subset C_c^\infty(K)$ that converges to ϕ in $\mathcal{D}(K)$, by continuity of F, we have $\langle F, \partial^\alpha \phi_j\rangle \to \langle F, \partial^\alpha \phi\rangle$ as $j \to \infty$. Hence $\partial^\alpha F|_{\mathcal{D}(K)}$ is continuous, and $\partial^\alpha F \in \mathcal{D}'(U)$.

The distribution $\partial^\alpha F$ is called the α^{th} *derivative of* F. Moreover, if $F_j \to F$ in $\mathcal{D}'(U)$, we have $\langle \partial^\alpha F_j, \phi\rangle = \langle F_j, \partial^\alpha \phi\rangle \to \langle F, \partial^\alpha \phi\rangle = \langle \partial^\alpha F, \phi\rangle$ for each $\phi \in C_c^\infty(U)$, and $\partial^\alpha F_j \to \partial^\alpha F$ in $\mathcal{D}'(U)$. Therefore, the differentiation operator $\partial^\alpha : \mathcal{D}'(U) \to \mathcal{D}'(U)$ is a continuous linear map with respect to the weak* topology.

In particular, for any locally integrable function $\psi \in L^1_{\text{loc}}(U)$, we can define its derivatives of arbitrary order even if it is not differentiable in the classical sense. To be specific, we define $\langle T_\psi, \phi\rangle = \int_U \psi(x)\phi(x)\,dx$. The derivative $\partial^\alpha T_\psi$ of the distribution T_ψ is called the α^{th} *distributional derivative of* ψ, denoted by $\partial^\alpha \psi$. Following are some examples of distributional derivatives.

Jump discontinuity. For simplicity, we first consider the functions on $\mathbb{R}$. Differentiating functions with jump discontinuities leads to δ-singularities. The simplest example is the *Heaviside step function* $H = \chi_{[0,\infty)}$, for which we have

$$\langle H', \phi\rangle = -\langle H, \phi'\rangle = -\int_0^\infty \phi'(x)\,dx = \phi(0) = \langle \delta, \phi\rangle, \quad \phi \in C_c^\infty(\mathbb{R}).$$

Hence the first distributional derivative of H is the Dirac function δ. More generally, for any $x \in \mathbb{R}$, the distributional derivative of the step function $\tau_x H = \chi_{[x,\infty)}$ is $\tau_x \delta$, which is the point mass at x.

If f is piecewise continuously differentiable on $\mathbb{R}$, f only has jump discontinuities at $x_1 < x_2 < \cdots < x_m$, and its pointwise derivative $\frac{df}{dx}$ is in $L^1_{\text{loc}}(\mathbb{R})$. Then

$$\begin{aligned}\langle f', \phi\rangle &= -\langle f, \phi'\rangle = -\sum_{j=0}^m \int_{x_j}^{x_{j+1}} f(x)\phi'(x)\,dx \\ &= -\sum_{j=0}^m \left[f(x_{j+1}^-)\phi(x_{j+1}) - f(x_j^+)\phi(x_j) - \int_{x_j}^{x_{j+1}} \frac{df}{dx}(y)\phi(y)\,dy\right] \\ &= \int_{-\infty}^\infty \frac{df}{dx}(y)\phi(y)\,dy + \sum_{j=1}^m \phi(x_j)\left[f(x_j^+) - f(x_j^-)\right]\end{aligned}$$

Therefore, the distributional derivative of f is given by

$$f' = \frac{df}{dx} + \sum_{j=1}^m \left[f(x_j^+) - f(x_j^-)\right]\tau_{x_j}\delta.$$

Generalized Heaviside step function.

3.3 Compactly Supported Distributions

The C^∞ topology. Let $U \subset \mathbb{R}^n$ be an open set. The C^∞ topology on the space $C^\infty(U)$ of all smooth functions on U is the topology of uniform convergence of functions, together with all their derivatives, on compact subsets of U. This topology can be defined by a countable family of seminorms as follows. Let (V_m) be an increasing sequence of precompact open subsets of U whose union is U. For each $m \in \mathbb{N}$ and each multi-index $\alpha \in \mathbb{N}_0^n$, define the seminorm

$$\|f\|_{[m,\alpha]} = \sup_{x\in \overline{V}_m} |\partial^\alpha f(x)|. \tag{3.3}$$

With the topology induced by the family of these seminorms, the space $C^\infty(U)$ is a Fréchet space. Furthermore, a sequence (f_j) converges to f in $C^\infty(U)$ if and only if $\|f_j - f\|_{[m,\alpha]} \to 0$ for all $m \in \mathbb{N}, \alpha \in \mathbb{N}_0^n$, if and only if $\partial^\alpha f_j \to \partial^\alpha f$ uniformly on compact sets for all $\alpha \in \mathbb{N}_0^n$.

Proposition 3.14. *Let $U \subset \mathbb{R}^n$ be an open set. The space $C_c^\infty(U)$ is dense in $C^\infty(U)$.*

Proof. We take the sequence (V_m) as in (3.3). By C^∞-Urysohn lemma [Theorem 1.10], for each m, we take $\psi_m \in C_c^\infty(U)$ with $\psi_m = 1$ on $\overline{V}_m$. If $\phi \in C^\infty(U)$, for all multi-indices $\alpha \in \mathbb{N}_0$., we have $\|\psi_m\phi - \phi\|_{[m_0,\alpha]} = 0$ for all indices $m \geq m_0$. Hence $\psi_m\phi \in C_c^\infty(U)$ converges to ϕ in the C^∞ topology. □

If U is an open subset of $\mathbb{R}^n$, we denote by $\mathcal{E}'(U)$ the space of all distributions on U whose support is a compact subset of U.

Theorem 3.15. *Let $U \subset \mathbb{R}^n$ be an open set.*

(i) If $F \in \mathcal{E}'(U)$, then F extends uniquely to a continuous linear functional on $C^\infty(U)$

(ii) If G is a continuous linear functional on $C^\infty(U)$, then $G|_{C_c^\infty(U)} \in \mathcal{E}'(U)$.

To summarize, $\mathcal{E}'(U)$ equals the dual space of $C^\infty(U)$.

Proof. If $F \in \mathcal{E}'(U)$, take $\psi \in C_c^\infty(U)$ such that $\psi = 1$ on $\operatorname{supp}(F)$, and define the linear functional G on $C^\infty(U)$ by $G\phi = \langle F, \psi\phi\rangle$. Since F is continuous on $\mathcal{D}(\operatorname{supp}(\psi))$, and the topology of the latter is defined by the norms $\phi \mapsto \|\partial^\alpha\phi\|_\infty$, there exists $C > 0$ and $N \in \mathbb{N}$ such that $|\langle G, \phi\rangle| = |\langle F, \psi\phi\rangle| \leq C\sum_{|\alpha|\leq N} \|\partial^\alpha(\psi\phi)\|_\infty$ for all $\phi \in C^\infty(U)$. By the product rule, if we choose m large enough so that $\overline{V}_m \supset \operatorname{supp}(\psi)$,

$$|\langle G, \phi\rangle| \leq C' \sum_{|\alpha|\leq N} \sup_{x\in\operatorname{supp}(\psi)} |\partial^\alpha\phi(x)| \leq C' \sum_{|\alpha|\leq N} \|\phi\|_{[m,\alpha]}.$$

Hence G is continuous on $C^\infty(U)$. By Proposition 3.14, the continuous extension G of F is unique.

On the other hand, if G is a continuous linear functional on $C^\infty(U)$, there exists constants C, m and N such that $|\langle G, \phi\rangle| \leq C\sum_{|\alpha|\leq N} \|\phi\|_{[m,\alpha]}$ for all $\phi \in C^\infty(U)$. Since $\|\phi\|_{[m,\alpha]} \leq \|\partial^\alpha\phi\|_\infty$, the functional G is continuous on $\mathcal{D}(K)$ for each compact $K \subset U$, and $G|_{C_c^\infty(U)} \in \mathcal{D}'(U)$. Moreover, if $\operatorname{supp}(\phi) \cap \overline{V}_m = \emptyset$, we have $\langle G, \phi\rangle = 0$, and $\operatorname{supp}(G) \subset \overline{V}_m$. Hence $G|_{C_c^\infty(U)} \in \mathcal{E}'(U)$. □

Remark. In fact, one can easily check that the operations of multiplication by C^∞ functions, translation, composition by invertible linear maps and differentiation, as is discussed in the last section, all preserves the class of $\mathcal{E}'(U)$. The case of convolution is a bit more complicated.

3.4 Tempered Distributions and Fourier Transform

Definition 3.16 (Tempered distributions)**.** A *tempered distribution (on* $\mathbb{R}^n$*)* is a continuous linear functional on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The space of tempered distribution is denoted by $\mathcal{S}'(\mathbb{R}^n)$. Usually, we equip $\mathcal{S}'(\mathbb{R}^n)$ with the weak* topology.

The following proposition helps to understand the relation of distributions and tempered distributions.

Proposition 3.17. *The space* $C_c^\infty(\mathbb{R}^n)$ *is dense in* $\mathcal{S}(\mathbb{R}^n)$*.*

Proof. We fix $\phi \in \mathcal{S}(\mathbb{R}^n)$, which is to be approximated. We take $\psi \in C_c^\infty(\mathbb{R}^n, [0,1])$ such that $\psi(0) = 1$, and let $\psi^t(x) = \psi(tx)$ for $t > 0$. Given any $N \in \mathbb{N}$ and $\epsilon > 0$, we can choose a compact $K \subset \mathbb{R}^n$ such that $(1+|x|)^N|\phi(x)| < \epsilon$ for all $x \notin K$. Then $\psi^t(x) \to 1$ uniformly on K as $t \to 0$, and

$$\lim_{t\to 0} \|\psi^t\phi - \phi\|_{(N,0)} \le \sup_{x\notin K}(1+|x|)^N|\psi^t(x)\phi(x) - \phi(x)| < \epsilon.$$

By arbitrariness of N and ϵ, we have $\|\psi^t\phi - \phi\|_{(N,0)} \to 0$ as $t \to 0$ for all $N \in \mathbb{N}_0$. For the terms involving derivatives, by the product rule,

$$(1+|x|)^N\partial^\alpha(\psi^t\phi - \phi) = (1+|x|)^N(\psi^t\partial^\alpha\phi - \partial^\alpha\phi) + R_t(x),$$

where the remainder R_t is a sum of terms involving derivatives of ψ^t. Since

$$\left|\partial^\beta\psi^t(x)\right| = t^{|\beta|}\left|\partial^\beta\psi(tx)\right| \le C_\beta t^{|\beta|},$$

we have $\|R_t\|_\infty \le Ct \to 0$ as $t \to 0^+$. An analogue of the preceding argument shows that $\|\psi^t\phi - \phi\|_{(N,\alpha)} \to 0$ as $t \to 0$. Hence $\psi^t\phi \in C_c^\infty(\mathbb{R}^n)$ converges to ϕ in $\mathcal{S}(\mathbb{R}^n)$, which completes the proof. ☐

Remark. Since the convergence in $\mathcal{D}(\mathbb{R}^n)$ implies the convergence in $\mathcal{S}(\mathbb{R}^n)$, if $F \in \mathcal{S}'(\mathbb{R}^n)$ is a tempered distribution, the restriction of F to $C_c^\infty(\mathbb{R}^n)$ is also continuous. Hence $F|_{C_c^\infty(\mathbb{R}^n)}$ is a distribution. Furthermore, by Proposition 3.17, the restriction $F|_{C_c^\infty(\mathbb{R}^n)}$ determines $F \in \mathcal{S}'(\mathbb{R}^n)$ uniquely. Thus we may identify $\mathcal{S}'(\mathbb{R}^n)$ with the sets of all distributions on $\mathbb{R}^n$ that extends continuously from $C_c^\infty(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

Example 3.18. Following are some examples of tempered distributions on $\mathbb{R}^n$.

- Every compactly supported distribution is tempered.
- If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n}(1+|x|)^{-N}|f(x)|\,dx < \infty$ for some $N \in \mathbb{N}_0$, then f is tempered, since

$$\left|\int_{\mathbb{R}^n} f(x)\phi(x)\,dx\right| \le \left\|(1+|x|)^{-N}f\right\|_{L^1}\left\|(1+|x|)^N\phi\right\|_\infty \le C\|\phi\|_{(N,0)}, \quad \phi \in \mathcal{S}(\mathbb{R}^n).$$

- Given $\omega \in \mathbb{R}^n$, the plane wave function $f(x) = e^{i\omega\cdot x}$ on $\mathbb{R}^n$ is a tempered distribution on $\mathbb{R}^n$. This distribution is related to the Fourier transform: if $\phi \in \mathcal{S}(\mathbb{R}^n)$, we have $\langle f, \phi\rangle = \widehat{\phi}(-\omega)$.
- In fact, the exponential function $f(x) = e^{\beta\cdot x}$ on $\mathbb{R}^n$ is tempered if and only if β is purely imaginary. We assume $\beta = \gamma + i\omega$ with $\delta, \omega \in \mathbb{R}^n$. If $\gamma \ne 0$, we take $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\psi(x)\,dx = 1$ and let $\phi_m(x) = e^{-\beta\cdot x}\psi(x - m\gamma)$. Then $\phi_m \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ as $m \to \infty$, but $\int_{\mathbb{R}^n} f\phi_m\,dx = \int_{\mathbb{R}^n}\psi\,dx = 1$.
- If $F \in \mathcal{S}'(\mathbb{R}^n)$, the derivative $\partial^\alpha F$ is also a tempered distribution. Indeed, $\phi_j \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ implies

$$\langle\partial^\alpha F, \phi_j\rangle = \langle F, \partial^\alpha\phi_j\rangle \to \langle F, \partial^\alpha\phi\rangle = \langle\partial^\alpha F, \phi\rangle.$$

- A function $\psi \in C^\infty(\mathbb{R}^n)$ is called *slowly increasing*, if ψ and all its derivatives have at most polynomial growth at infinity, i.e. for every multi-index α there exists $N_\alpha \in \mathbb{N}_0$ such that $|\partial^\alpha\psi(x)| \le C_\alpha(1+|x|)^{N_\alpha}$. If $F \in \mathcal{S}'(\mathbb{R}^n)$, the product ψF with a slowly increasing C^∞ function is also a tempered distribution.
- Let $F \in \mathcal{S}'(\mathbb{R}^n)$. If $y \in \mathbb{R}^n$, the translated distribution $\tau_y F$ is also tempered; If T is an invertible linear mapping on $\mathbb{R}^n$, the composition $F \circ T$ with an invertible linear map is also tempered.

Proposition 3.19. *If $F \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, the function $(F * \psi)(x) = \langle F, \tau_x \widetilde{\psi} \rangle$ is a slowly increasing C^∞ function, and we have*

$$\langle F, \phi * \widetilde{\psi} \rangle = \int_{\mathbb{R}^n} (F * \psi)(x) \phi(x) \, dx, \quad \phi \in \mathcal{S}(\mathbb{R}^n)..$$

Proof. □